MATH 401 INTRODUCTION TO ANALYSIS-I,
SPRING TERM 2024, PRACTICE EXAM 2

Note that the second exam is on Wednesday 13th March, at 1:25 in Room 011 Huck.

1. Let $\mathcal{A}$, $\mathcal{B}$ be non–empty sets of real numbers which are bounded above, and let $\mathcal{A} + 2\mathcal{B}$ denote the set of numbers of the form $a + 2b$ with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.
(i) Prove that $\sup(\mathcal{A} + 2\mathcal{B})$ exists.
(ii) Prove that $\sup(\mathcal{A} + 2\mathcal{B}) \leq \sup \mathcal{A} + 2 \sup \mathcal{B}$.
(iii) Let $\delta > 0$. Prove that there are $a \in \mathcal{A}$ and $b \in \mathcal{B}$ such that $a > \sup \mathcal{A} - \delta$ and $b > \sup \mathcal{B} - \delta$.
(iv) Deduce that $\sup(\mathcal{A} + 2\mathcal{B}) = \sup \mathcal{A} + 2 \sup B$.

2. Let $\mathcal{A} = \left\{2 + \frac{3}{\sqrt{n}} : n \in \mathbb{N}\right\}$. Prove that $\inf \mathcal{A}$ exists and $\inf \mathcal{A} = 2$.

3. Prove, using only the definition of a limit, that

$$\lim_{n\to\infty} \frac{n^3 + n}{n^3 + n + 7} = 1.$$

4. Suppose that $\lim_{n\to\infty} a_n = l$ and $l > 2$. Prove that there is an N such that whenever $n > N$ we have $a_n > 2$.