On the Diophantine equation $a^3 + b^3 + c^3 + d^3 = 0$

by

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Introduction. The equation

$$a^3 + b^3 + c^3 + d^3 = 0 \tag{1}$$

has been studied by many mathematicians since Diophantus (see [B, p. 24]; [C]; [D, pp. 550–562]; [S]). Partial solutions in integers and complete solutions in rational numbers have been found.

A general solution of (1) found by Euler (see [H, pp. 290–291) involves the following polynomials $s_i = s_i(x, y, z)$:

$s_1 := 9x^3 + 9x^2y + 3xy^2 + 3y^3 - 3x^2z + 6xyz - 3y^2z + 3xz^2 + yz^2 - z^3$,

$s_2 := 9x^3 - 9x^2y + 3xy^2 - 3y^3 + 3x^2z + 6xyz + 3y^2z + 3xz^2 - yz^2 + z^3$,

$s_3 := -9x^3 + 9x^2y - 3xy^2 + 3y^3 + 3x^2z + 6xyz + 3y^2z - 3xz^2 + yz^2 + z^3$,

$s_4 =: -9x^3 - 9x^2y - 3xy^2 - 3y^3 - 3x^2z + 6xyz - 3y^2z - 3xz^2 - yz^2 - z^3$.

Using these polynomials, all rational solutions to (1) can be described in the following way:

Theorem 1. *All rational solutions* (a, b, c, d) *of* (1) *up to nonzero rational factors are in* 1-1 *correspondence with all triples* (x, y, z) *up to nonzero rational factors according to the formulas* $(x, y, z) \mapsto (a, b, c, d) = (s_1, s_2, s_3, s_4)$; *and* $(a, b, c, d) \mapsto (x, y, z)$
$= (ac - bd, -a^2 + ab - b^2 + c^2 - cd + d^2, a^2 - ab + b^2 - ac + 2bc + c^2 + 2ad - bd - cd + d^2)$.

This theorem allows us to describe all integer solutions as follows:

Corollary 2. *Up to a rational factor, every integral solution* (a, b, c, d) *of* (1) *is equal to* (s_1, s_2, s_3, s_4) *where* $s_i = s_i(x, y, z)$ *are as above with integers* x, y, z. *Every integral primitive solution* (a, b, c, d) *of* (1) *can be written uniquely as* $(s_1, s_2, s_3, s_4)/D$ *with* $D = \gcd(s_1, s_2, s_3, s_4)$ *where* s_i *are as above with integral primitive* (x, y, z).

Recall that an n-tuple $(u_1, \ldots, u_n)$ of integers is called *primitive* if $\gcd(u_1, \ldots, u_n) = 1$. Here gcd stands for the greatest common divisor which takes integral nonnegative values.

Corollary 2 does not give an explicit description of all integral primitive solutions to (1) since it is not clear from the definition $D = \gcd(s_1, s_2, s_3, s_4)$ what are possible values for D when (x, y, z) ranges over all integral primitive triples. So the complete solution of (1) in integers was pointed out as an open problem in [B, p. 10], [[C, p. 1251], [H, p. 290], [Ha, pp. 199–200], [R, v.3, p. 197], [S, pp. 121–122].

For comparison, let us consider a simpler Diophantine equation

$$a^2 + b^2 = c^2. \tag{2}$$

It is well-known that every primitive integral solution (a, b, c) of (2) can be written as $\pm(x^2 - y^2, 2xy, x^2 + y^2)/d$ with a primitive pair (x, y) which is unique, up to sign, and

$$d = \gcd(x^2 - y^2, 2xy, x^2 + y^2).$$

The explicit description of this number d is given as follows: $d = \mathrm{mod}(x,\,2) + \mathrm{mod}(y,\,2)$. Here and further on, $\mathrm{mod}(x,\,2)$ denotes the remainder on division of x by 2. So $d = 2$ when xy is odd, and $d = 1$ otherwise. Thus, the equation (2) is completely solved in integers.

In this paper we address the problem of finding an explicit description of the number D in Corollary 3. Our main result is:

Theorem 3. *Every integral primitive solution (a, b, c, d) of (1) can be written uniquely as $(s_1, s_2, s_3, s_4)/D$ with $D = \gcd(s_1, s_2, s_3, s_4) = d_0 d_2 d_3$ where polynomials $s_i = s_i(x, y, z)$ are as above with integral primitive (x, y, z) and*

$$d_0 = \gcd(x, 3y^2 + z^2)\gcd(y, 3x^2 + z^2)\gcd(z, 3x^2 + y^2),$$

$$d_2 = \begin{cases} 4 & \text{when } \mathrm{mod}(x,2)+\mathrm{mod}(y,2)+\mathrm{mod}(z,2) = 2 \text{ and } \mathrm{mod}(xyz,4) \neq 0, \\ 2 & \text{when } \mathrm{mod}(x,2)+\mathrm{mod}(y,2)+\mathrm{mod}(z,2) = 2 \text{ and } \mathrm{mod}(xyz,4) = 0, \\ 1 & \text{otherwise,} \end{cases}$$

$$d_3 = \begin{cases} 3 & \text{when } \mathrm{mod}(z,3) = 0 \text{ and } \mathrm{mod}(xy,3) \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Our description of D is explicit enough to answer the question of what are its possible values.

Corollary 4. *When (x, y, z) ranges over all primitive integral triples, the number D in Corollary 2 ranges over all numbers of the form $t_2 t_3 t$ where t_2 is 1 or 8, t_3 is 1, 3, or 9, and t is any product of primes of the form $3k + 1$.*

Note that while the number d for the equation (2) is bounded, the number D for the equation (1) is not. While there are many polynomial solutions for (1) with integral coefficients besides (s_1, s_2, s_3, s_4), we believe that the set of primitive integral solutions of (1) cannot be covered by a finite set of polynomial families with integral coefficients.

1. **Proof of Theorem 3.** By Theorem 1, $(s_1, s_2, s_3, s_4) = 0$ if and only if $(x, y, z) = 0$. Assume now that $(s_1, s_2, s_3, s_4) \neq 0$. Let $D = \gcd(s_1, s_2, s_3, s_4)$. Let p^m be a primary factor of D, i.e., p is a prime and p^m is the highest power of p dividing D.

We have to prove that p^m is equal to the highest power p^n of p dividing $d_0 d_2 d_3$. (By definition, D and all d_i are positive.)

Now we consider three cases:

Case $1 : p \geq 5$. First we prove that $m \leq n$, i.e., $p^m | d_0$. If $m = 0$, there is nothing to prove, so let $m \geq 1$.

Since $p^m | s_1$, we conclude: if p divides both x and y, then $p | z^3$, hence p divides z; if p divides both x and z, then p divides $3y^3$; if p divides both z and y, then p divides $9x^3$. Since (x, y, z) is primitive and $p \neq 3$, p divides at most one of the numbers x, y, z.

Since p^m divides $s_1 + s_2 + s_3 + s_4 = 24xyz$, we conclude that p^m divides x, y or z.

Subcase $1x : p \geq 5$ and p^m divides x. Then p^m divides

$$(s_2 + s_3)|_{x=0} = 2z(3y^2 + z^2)$$

hence $p^m | 3y^2 + z^2$. Therefore $p^m | d_0$.

Subcase $1y : p \geq 5$ and p^m divides y. Then p^m divides

$$(s_2 + s_3)|_{y=0} = 2z(3x^2 + z^2)$$

hence $p^m | 3x^2 + z^2$. Therefore $p^m | d_0$.

Subcase $1z : p \geq 5$ and p^m divides z. Then p^m divides

$$(s_1 + s_2)|_{z=0} = 6z(3x^2 + y^2)$$

hence $p^m | 3x^2 + y^2$. Therefore $p^m | d_0$.

Thus, $p^m | d_0$ in all three subcases, i.e., $m \leq n$ in Case 1.

Since $p^n | d_0$, we conclude that either p^n divides both x and $3y^2 + z^2$, or it divides both y and $3z^2 + x^2$ or else it divides both z and $3x^2 + y^2$. Therefore in all three cases it divides D, i.e., $m \geq n$. Thus, $m = n$ in Case 1.

Case $2 : p = 2$. When only one or all three of (x, y, z) are odd, D is odd, so $m = 0$. In this case, also all three d_0, d_2, d_3 are odd so $n = 0 = m$.

Assume now that exactly 2 of the numbers (x, y, z) are odd, i.e.,

$$\mathrm{mod}(x, 2) + \mathrm{mod}(y, 2) + \mathrm{mod}(z, 2) = 2.$$

Then $m = 3$, and $n = 3$ as well.

Case $3 : p = 3$. When $\mathrm{mod}(z, 3) \neq 0$, $m = n = 0$. When $\mathrm{mod}(z, 3) = 0$ and $\mathrm{mod}(y, 3) \neq 0$, then $m = 1$, and also $n = 1$. When $\mathrm{mod}(z, 3) = 0 = \mathrm{mod}(y, 3)$, then $m = 2$, and also $n = 2$.

2. **Proof of Corollary 4.** In the proof of Theorem 3 in Cases 2 and 3, we saw that the power of 2 dividing D can be 1 or 8, and the power of three dividing D can be 1, 3 or 9. Now let p be a prime ≥ 5 dividing D. Modulo such a prime, -3 is a square, therefore the multiplicative group modulo p contains an element of order 3, so $p - 1$ is divisible by 3, i.e., $p = 3k + 1$. Therefore D has the form described in Corollary 6.

Now it remains to be proved that every number D of this form can actually occur. We write $D = 2^m 3^n d'$ where $m = 0$ or 3, $n = 0$, 1, or 2, and d' is a product of primes of the form $3k + 1$.

We take $x = d'$. We set

$$y = \begin{cases} 1 & \text{when } n \leq 1, \\ 3 & \text{otherwise.} \end{cases}$$

Note that -3 is a square modulo every prime of the form $3k + 1$, and so is $-3y^2$. By the Chinese remainder theorem, we can choose z such that $\mathrm{mod}(z^2 + 3y^2, x) = 0$, i.e., $\gcd(x, 3y^2 + z^2) = x = d'$. In addition, we can impose on z any congruences modulo 4 and 3. We require z to be odd when $m = 0$ and $\mathrm{mod}(z, 4) = 0$ otherwise. We require $\mathrm{mod}(z, 3) =$

mod$(D,3)$. Still we can replace z by $z+12ux$ with any integer u keeping all these conditions intact. Since $\gcd(3x^2+y^2,12x)$ divides 12, we can arrange that $\gcd(z,3x^2+y^2)|12$.

Then:

$$\gcd(z,3x^2+y^2)=\begin{cases}1 & \text{when } n\le 1 \text{ and } m=0,\\ 3 & \text{when } n=2 \text{ and } m=0,\\ 4 & \text{when } n\le 1 \text{ and } m\ge 1,\\ 12 & \text{otherwise,}\end{cases}$$

$$\gcd(x,3y^2+z^2)=d',$$

and

$$\gcd(y,3x^2+z^2)=\begin{cases}1 & \text{when } n\le 1,\\ 3 & \text{otherwise.}\end{cases}$$

Therefore $d_0=2^{m'}3^{n'}d'$ where

$$m'=\begin{cases}0 & \text{when } m=0,\\ 2 & \text{when } m=3,\end{cases}$$

and

$$n'=\begin{cases}0 & \text{when } n\le 1,\\ 2 & \text{when } n=2,\end{cases}$$

hence $D=d_2d_3d'$.

References

[B] B. C. Berndt, Y.-S. Choi, and S.-Y. Kang, *The problems submitted by Ramanujan to the Journal of the Indian Mathematical Society*, in: Proceedings of conference, “Continued Fractions: From Analytic Number Theory to Constructive Approximation,” held at the University of Missouri in May, 1998; B. C. Berndt and F. Gesztesy, editors, AMS Cont.Math. 236 (1999), 15–56.

[C] A. Choudhry, *On equal sums of cubes*, Rocky Mountain Journal of Math. 28:4 (1998), 1251–1257.

[D] L.E. Dickson, *History of the theory of numbers*, New York, Chelsea Pub. Co., (1966), v. 2. Diophantine analysis.

[H] L.-K. Hua, *Introduction to number theory*, Berlin; New York, Springer-Verlag, 1982.

[Ha] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 5th ed. Oxford, Clarendon Press; New York, Oxford University Press, 1979.

[R] Ramanujan’s notebooks, edited by B C. Berndt, New York, Springer-Verlag, 1985–1998, 5 v.

[S] C. Sándor, *On the equation* $a^3+b^3+c^3=d^3$, Period. Math. Hungar. 33:2 (1996), 121–134.

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