Math 436. Midterm 3. Nov. 25, , 2002. 5 problems, 10 pts each

Name

---

1. For which $t$ the matrix $\begin{pmatrix} 4 & t \\ t & 1 \end{pmatrix}$ is positive definite?

Solution. Fot $t$ such that $4 > 0, 4 - t^2 > 0$, i.e., $-2 < t < 2$.

2. Determine the definiteness for the quadratic form $(x+y+z)^2 + xy + yz + xz$.

Solution. The matrix is $\begin{pmatrix} 1 & 3/2 & 3/2 \\ 3/2 & 1 & 3/2 \\ 3/2 & 3/2 & 1 \end{pmatrix}$. The eigenvalues are $-1/2, -1/2, 4$. The quadratic form is indefinite.

3. Find the characteristic polynomial and eigenvalues of the $3 \times 3$ matrix with all entries equal to $10^{-10}$.

Solution. The eigenvalues are $0, 0, 3 \cdot 10^{-10}$ with eigenvectors
$(1, -1, 0, 0, )^T, (0, 1, -1, 0, )^T, (1, 1, 1, 1)^T$.
Remark: For any nonzero number $c$, eigenvalues of $cA = c$ (eigenvalues of $A$) and eigenvvectors of $cA =$ eigenvvectors of $A$.

4. Find SVD for the matrix $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Solution. $A = U \begin{pmatrix} \sqrt{14} \\ 0 \\ 0 \end{pmatrix}$ with an orthogonal matrix $U$. The first row of $U$ is $A/\sqrt{14}$. It is easy to find two linearly independent vectors orthogonal to $A$, e.g., $u = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$. The vector $v - 1.2u$ is orthogonal to both $A$ and $u$. The vectors $A/\sqrt{14}, u/\sqrt{5}, (v - 1.2u)/\sqrt{70}$ form an orthonormal basis. So we can take them as the columns of $U$.

5. Compute $(A - 1_3)(A - 2 \cdot 1_3)(A - 3 \cdot 1_3)$ for the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$.

Solution. We know that $A = CBC^{-1}$ for $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ (diagonal matrix). For any polynomial $p(x)$ we have $p(A) = Cp(B)C^{-1}$. For our polynomial
$p(x) = (x-1)(x-2)(x-3)$, we have $p(B) = 0$. So $p(A) = 0$.

Remark: Similarly, any matrix with distinct eigenvalues satisfies its characteristic equation. By additional work, we can show that every matrix satisfies its characteristic equation.