Math 436. Midterm 2. October 30, 2002. 5 problems, 10 pts each

Name Leon Vaserstein

1. Solve for a

$a(x^2-1) = x^3 - 1$

where x is a given number.

Solution. If $x \neq \pm 1$, we have exactly one solution $a = (x^2+x+1)/(x+1)$. If $x = 1$, the equation is $0 = 0$ and every a is a solution If $x = -1$, the equation is $0 = -2$ and there are no solutions.

2. Find the least-squares solution x to

$x = a, x = b, x = c,$

where a, b, c are given numbers.

Solution. In matrix form, our system is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

It has no solutions unless $a = b = c$. We multiply both sides by the transpose $(1,1,1)$ of the coefficient matrix to get a system for the least-squares solutions:

$$3x = a + b + c.$$

So $x = (a+b+c)/3$.

3. Compute the trace and determinant of the matrix $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 & 0 \\ 0 & 0 & 6 & 0 & 0 \end{pmatrix}$.

Solution. $\mathrm{tr}(A) = 1+7+1+5+0 = 14$. $\det(A) = \det(\begin{pmatrix} 1 & 2 \\ 6 & 7 \end{pmatrix}) \det(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{pmatrix})$ with the first determinant, $\det(\begin{pmatrix} 1 & 2 \\ 6 & 7 \end{pmatrix} = 7-12 = -5$. To compute the second determinant, we switch the 1st and 3nd column and obtain am upper triangular matrix whose determinant is $3 \cdot 5 \cdot 6 = 90$. So $\det(A) = (-5)(-90) = 450$.

4. Compute A^2 and $A^T A$ for matrix A in Problem 3.

Solution. $A^2 = \begin{pmatrix} 13 & 16 & 68 & 30 & 14 \\ 48 & 61 & 82 & 40 & 54 \\ 0 & 0 & 27 & 12 & 3 \\ 0 & 0 & 24 & 33 & 12 \\ 0 & 0 & 6 & 12 & 18 \end{pmatrix}$ and $A^T A = \begin{pmatrix} 37 & 44 & 51 & 4 & 5 \\ 44 & 53 & 62 & 8 & 10 \\ 51 & 62 & 126 & 34 & 18 \\ 4 & 8 & 34 & 45 & 26 \\ 5 & 10 & 18 & 26 & 34 \end{pmatrix}$.

5. Find the eigenvalues and eigenvectors for the linear transformation
$f(x) \mapsto f'(x) + f(x-1) + f(x)$
on the space of the poynomials $f(x)$ in x of degree ≤ 2.

Solution. We uses the basis $1, x, x^2$. under the transformation, $1 \mapsto 2, x \mapsto 2x, x^2 \mapsto 2x + (x-1)^2 + x^2 = 2x^2 + 1$. The matrix of the transformation is $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. The eigenvalues of this upper trizangular matrix are the diagonal entries 2, 2, 2. The egenvectors are nonzero vectors in the kernel of $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The kenel has dimension 2, with a basis consisting of the fist two basic vectors, 1 and x. So the eigenvectors are the nonzero polynomials of degree ≤ 1.