Technology Diffusion by Learning from Neighbours

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Abstract

In this paper, we consider a model of social learning in a population of myopic, memoryless agents. The agents are placed on integer points on an infinite line. Each period, they perform experiments with one of two technologies, then each observes the outcomes and technology choices of the two adjacent agents as well as his own outcome. Two learning rules are considered; it is shown that under the first, where an agent changes his technology only if he has had a failure (a bad outcome), the society converges with probability 1 to the better technology. In the other, where agents switch on the basis of the neighbourhood averages, convergence occurs if the better technology is sufficiently better. The results provide a surprisingly optimistic conclusion about the diffusion of the better technology through imitation, even under the assumption of extremely boundedly rational agents.

1 Introduction

This paper considers a problem of social learning with two distinctive features. First, the agents we model are minimally rational; they do not look ahead and do not do Bayesian revision of probabilities, nor do they remember outcomes from the past. Second, they observe only local information-the outcomes of their own experiments in each period and those of their immediate neighbours and they have the ability to compare averages. The problem we seek to examine is whether, despite these disadvantages, players can all learn to use the superior one of two available technologies.

Similar issues have recently been considered in models of cultural evolution (Bisin and Verdier(2001) for example) and in models of learning by boundedly rational agents (such as the paper of Ellison and Fudenberg (1995) on word-of-mouth learning, which is discussed at length later on, or of Eshel, Samuelson and Shaked (1998).) Cultural evolution stresses the role of propagation of cultural traits by horizontal transmission or imitation. Here the larger the proportion of individuals with a particular trait, the more likely the trait is to spread, thus creating a pressure towards conformity. The Eshel, Samuelson and Shaked paper adds another dimension to this problem, because individuals change their behaviour not just to conform to the majority view but if they see alternative strategies yielding a greater payoff on average among other agents they are able to observe. Their paper considers a finite number of agents placed in a circle, with each agent able to observe his or her neighbours; the authors obtain a surprisingly optimistic result on the survival of altruism. Of course, an individual altruist might be doing very poorly, but he or she will not change because her neighbouring altruists are doing very well.

We consider in this paper a problem of diffusion of technology, where one technology is better than the other and agents imitate better technologies among their neighbours. This could be interpreted as a model of cultural evolution, since it is based on imitation. As stated earlier, we impose minimal rationality requirements on our agents, so one might regard them as proto-humans or automata rather than actual humans.

Specifically, agents are distributed at integer points on the line. Suppose there are two available technologies, $B(lue)$ and $R(ed)$. Each agent on the (infinite in both directions) line is randomly assigned a technology; that is, each integer site on the line is B or R with a strictly positive probability (for example 1/2). Each agent, labelled henceforth as agent i, $i = ... -$

$3, -2, -1, 0, 1, 2, 3...$, uses his assigned technology in each period to produce an output, which could either be 1(Success) or 0(Failure).[1] The R technology is better than the B technology in the sense that $p_B < p_R$, where p_B and p_R are the probabilities of success with the blue and red technologies respectively.

In any period t, the agents simultaneously and independently perform the experiment with their assigned technologies. Each agent observes his own outcome and the outcomes of his two nearest neighbours. Agents are automata but can learn from one another according to one of the two learning rules to be described. In rule L_1, the agent never changes technology if his own outcome is a success. If his outcome in a given period t is a failure, he considers himself (site i) as well as sites $(i-1, i+1)$. If he is using B and the agents using R have a greater *proportion* of successes in t than those using B, he switches next period to using R. The process of switching from R to B is analogous.[2] In rule L_2, agent i always looks around, calculates whether the other technology has a higher proportion of successes among agents $\{i-1, i, i+1\}$ and switches to that technology next period, if this is in fact the case. The object of this paper is to investigate the question of whether the better technology diffuses through the population, even though agents only consider current results and do not behave optimally. (This problem, with discounting, is a two-armed bandit problem, where optimal behaviour has been much studied in the literature.)

The main result we obtain is as follows: With rule L_1, from any initial configuration that has positive probability of occurrence, the better technology diffuses across the entire population with probability 1. With rule L_2, we have only been able to prove a weaker result, namely that the better technology diffuses with probability 1 from any positive probability initial configuration, if p_R is sufficiently greater than p_B (in a precise way).

While we do not claim that this exactly models some real-world phenomenon in the economy, we argue that the model has aspects that reflect the important questions raised earlier in the introduction. Individuals do learn from their neighbours and better ways of doing things (or "technolo-

[1] If the payoffs are real-valued, as in some of the other papers in the literature, one could think of "Success" as being a payoff greater than some arbitrary aspiration level, and "Failure" as being a payoff below this level. The conditions on the probabilities of success and failure imposed here can then be translated into conditions on the probability distribution of output for R and B; for learning rule L_1, stochastic dominance of R over B is sufficient to generate the results.

[2] This avoids the problem of the agent not giving any special attention to his own payoff.

gies") do spread as a result. It is by no means clear in advance whether the better technology always prevails, whether either could prevail (as is the case with a finite number of agents) or whether the technologies co-exist (as happens frequently in developing countries).

While we have not come across this specific model anywhere, there are related problems that have been considered in at least three different fields, namely physics, probability and the analysis of learning models in economics. In physics, the problem would formally fall in the category of probabilistic cellular automata. A paper by Bhargava, Kumar and Mukherjee (1993) in fact describes a model of new-product diffusion in these terms. (The authors are physicists.) In their model, there is one innovation and each agent is in one of two states–an adopter or non-adopter. Once someone has adopted, he or she never switches to becoming a non-adopter. Any non-adopter with an adopter as a neighbour (the neighbourhood is a two-dimensional chessboard type with eight neighbours), becomes an adopter with probability $1 - x_t$, where $x_t \to 1$ as $t \to \infty$. The authors simulate a 100×100 grid with different seed values of adopters and find, in their simulations, that the initial value of x is crucial in determining the rate at which adoption takes place (they specify in advance how x increases with time) and that the initial number of adopters ceases to have an effect beyond a certain level. It is clear that our model is somewhat different even though the question addressed is similar. Mehta and Luck (1999) consider the statistical mechanical properties of a two-dimensional system motivated, in part, by conversations we have had on the model of this paper, though they incorporate several differences, including a "biasing" parameter that specifies how quickly agents using each technology (in our language) are ready to change; changes of technology also do not happen every period, another parameter is the extent of inertia for each type. Their analysis is done using an approximation technique called "pair approximation" and by simulation, and their results predict co-existence or clustering depending on the different parameter values. Our model is simpler, does not involve biasing parameters and we obtain exact results.

The analogous literature in probability theory is both extensive and deep and we will not survey it here, except to state that the usual modelling framework in this literature has the technology changing as a result of the proportion of neighbours using the other technology rather than as the proportion of successes (see,for example, the voter model, where either technology can prevail with probability 1). While most of the work in "interacting particle systems" is in continuous time, Durrett (1988) also discusses some discrete-time models.

The probabilistic techniques we use are considerably less sophisticated than the ones employed in the literature above, though we do use the method of coupling that is also common in probability work. (See, for example, Chapter 14 in Aldous and Fill (2002) or Thorisson (2001) or Ross(1996).)

As alluded to earlier, there is a considerable literature in economics both on diffusion of innovations and on learning from others. In the latter category are the papers of Anderlini and Ianni (1996) and (1997), Bala and Goyal (1998), Banerjee and Fudenberg (2000), Ellison (1993), Ellison and Fudenberg (1993), Ellison and Fudenberg (1995) and Morris (2000). (This is by no means an exhaustive list.) We discuss some of these papers in more detail below.

Binmore and Samuelson (1997) is representative of a number of papers that use "aspiration-based" switching of strategies.This paper has the following "learning" model. A player gets a "learn-draw" at some point in time and only that player has an opportunity to learn (this is sometimes referred to as asynchronous updating). He/she then compares his realised payoff with his current strategy to an aspiration level (this is analogous to a success in our learning rule L_1). If it is greater he does nothing. If it is less than the aspiration level, he chooses a random player from the (finite) population and imitates him with probability 1-λ; with probability λ, he chooses the strategy that the randomly chosen player is not playing. Our paper differs in the following respects: (1) In our paper, updating occurs for everyone every period. (2) The population is infinite. (3) The probability that the neighbour is imitated is not exogenous in our model; it depends on the performance of the two strategies in the neighbourhood of the agent who is considering changing.

Bala and Goyal (1998), whose paper has a very similar title to ours, in fact have a very different model. In their model, agents located at nodes in an infinite, connected graph choose an action with uncertain payoff. Though they are not completely rational, being myopic maximisers, they are rational learners in the sense of using Bayesian updating every period, so that all past experience is in fact taken into account. Our agents are memoryless and follow heuristics rather than maximising posterior expected utility. Bala and Goyal show convergence to the optimal action, given the updating rules and connectedness, for sufficiently dispersed prior probabilities. Of course, this assumption has no counterpart in our (non-Bayesian) model.

The paper closest to ours in conception, though very different in execution, is Ellison and Fudenberg (1995). They consider a continuum of identical players, each of whom chooses in each period the technology he or she will use in that period. The payoff of each technology is subject to stochastic shocks. A fraction of players keep using the same technology in the next period, but the complementary fraction "hears" of the experience of N randomly drawn other players and each member of this f(r)action switches to the technology with the higher average payoff, so long as both technologies are present in the sample. (Except for the agents that do not change, this learning rule is like our L_2.) The state variable in this model is the aggregate proportion of individuals using the better technology. Ellison and Fudenberg study a local linearisation of this dynamical system and, in their theorem 2, characterise the conditions under which efficient social learning (of the better technology) will take place. In contrast to their paper, word-of-mouth in our paper is local (there is an explicit neighbourhood structure). The difference this causes might be thought of by interpreting their paper as one in which each player has N neighbours, whenever it is that player's turn to move (learn) and the neighbours change independently to a non-overlapping set of N each time the player moves. Thus while, if the player concerned learns, his or her choice depends on the neighbours, the choices made by the neighbours do not depend on his. (This is assuming the difference in payoffs of the technologies is constant. Ellison and Fudenberg actually vary this as well, with a "shock" common to all players, so agents' choices are in fact correlated.) Our results are therefore somewhat different from theirs, and we are able to look explicitly at the dynamics of the total configuration (an agent is at each site, and a configuration consists of a technology for each site), rather than at the aggregate proportion.

In an earlier, related paper, Ellison and Fudenberg (1993) discuss social learning. In the first part of their paper, which corresponds broadly to ours, the same technology is better for everyone. There is an initial proportion of people using this technology, with everyone being concentrated at one site. Players in each period observe the payoffs of all available technologies in the previous period. Only an exogenously given proportion α are ready to learn, that is to change the technology being used to the one with higher payoff in the previous period. While Ellison and Fudenberg show that this learning process need not converge to the best technology, a sufficient degree of "popularity weighting" will lead to such a convergence. Popularity weighting is a reflection of past experience with each technology, since it uses the information of how popular a given technology is in addition to its payoff the previous period. Popularity weighting plays no role in our paper; agents know only their neighbours' realisations in each period. (In Ellison and Fudenberg, realisations are perfectly correlated so one can interpret their paper in our terms as each player always having access to the payoffs of both technologies in the previous period.)

The other papers are somewhat less related in conception, since they are concerned either with rational learning or with co-ordination games; however the papers of Anderlini and Ianni, Ellison and Morris do consider local interaction though in settings different from that of this paper. Our results cannot therefore be directly compared to theirs, except in the method of analysis. We use random walk and coupling arguments, which are somewhat simpler than the ones in these other papers.

2 Technology Diffusion under Rule L_1

In this section, we show that if initially an agent at an integer site is independently assigned R with a strictly positive probability, and if R is more likely to yield a success than B does, then under rule L_1 every agent will use R eventually with probability 1.

Our approach to establish the convergence result can be outlined as follows. First we consider a special configuration, denoted by *Config*-$[\ell_0, r_0]$, $\ell_0 < r_0$, which initially has a single segment of consecutive red agents located at the integer points of interval $[\ell_0, r_0]$. We show that, with a strictly positive probability, Config-$[\ell_0, r_0]$ will become all red before it is either absorbed by all blue or becomes a single red site. We then repeatedly couple the original process with a sequence of such configurations, Config-$[\ell_0^j, \ell_0^j + 1]$, $j = 1, 2, \ldots$, so that as soon as one of those configurations is absorbed by all red, which occurs almost surely, the original process is also absorbed by all red.

The details are given in the following two subsections.

2.1 Config–$[\ell_0, r_0]$

This section studies a special configuration that initially has a single segment of i consecutive red sites and all other sites blue (the locations of those red sites are immaterial since the line extends to infinity in both directions). We show that such a configuration has the property that before it becomes either all red or all blue, the two absorbing states, it always contains a single segment of consecutive red sites. We then show that for $i \geq 2$, the configuration has a positive probability to become all red.

For convenience, we call a site a R-site (a B-site) in period t if the agent located at that site uses R (B). A segment of consecutive R-sites (B-sites) will be called a R-interval (a B-interval). We say a configuration in period t is in state $x_t = [\ell_t, r_t]$, $\ell_t \leq r_t$, if the entire line is blue except the R-interval $[\ell_t, r_t]$. Let $|x_t| = r_t - \ell_t + 1$ be the cardinality of x_t, which is understood as the number of R-sites in interval $[\ell_t, r_t]$, $t = 0, 1, \ldots$. We also say the configuration is in state $\mathcal{B}$ ($\mathcal{R}$) if the entire line is blue (red). Here and in the sequel, the configuration with initial state $[\ell_0, r_0]$ is called *Config*-$[\ell_0, r_0]$.

Under learning rule L_1, if the outcome of agent i's chosen technology in period t is a success, then he uses the same technology in period $t+1$. If the outcome of his experiment in period t is a failure, and if at least one of his neighboring sites using the alternative technology is a success, then agent i switches to the alternative technology in period $t+1$. More specifically, the switching probabilities of the central agent i under rule L_1 are as follows:

1. BBB or RRR: No switch
2. BBR or RBB: Switch with probability $(1 - p_B)p_R$;
3. RRB or BRR: Switch with probability $(1 - p_R)p_B$;
4. BRB: Switch with probability $(1 - p_R)(1 - (1 - p_B)^2)$;
5. RBR: Switch with probability $(1 - p_B)(1 - (1 - p_R)^2)$.

It is worth noting that, if the neighbours of the central agents use different technologies, such as in cases (2) and (3), then the central agent, say B in BBR, only needs to learn from his right neighbouring site R, and switches to R in period $t + 1$ if B is a failure and R is a success in period t.

Our first lemma shows that under rule L_1, Config-$[\ell_0, r_0]$, a configuration with initial state $X_0 = [\ell_0, r_0]$, $\infty < \ell_0 \leq r_0 < \infty$, in period t must be either in state $\mathcal{B}$ or in some random state $X_t = [L_t, R_t]$, $L_t \leq R_t$, $t = 0, 1, \ldots$. In other words, a configuration starting with a single R-interval always consists of a single R-interval as time evolves until absorption by $\mathcal{B}$, if that event ever occurs.

Lemma 1 *If $\ell_0 \leq r_0$, then under rule L_1 Config-$[\ell_0, r_0]$ must be either in state $\mathcal{B}$ or in state $X_t = [L_t, R_t]$, for some $L_t \leq R_t$, in period t, $t = 0, 1, \ldots$.*

Proof. The statement is trivially true for $t = 0$. If in period t Config-$[\ell_0, r_0]$ is in state $\mathcal{B}$, then clearly it remains in state $\mathcal{B}$ in period $t+1$. Let us assume that in period t Config-$[\ell_0, r_0]$ is in state $x_t = [\ell_t, r_t]$ for some $\ell_t \leq r_t$, where x_t is a possible realization of $X_t = [L_t, R_t]$. We show that in period $t+1$ Config-$[\ell_0, r_0]$ will either be absorbed by $\mathcal{B}$ or consist of a single R-interval. We need to examine two cases, depending on the value of $|x_t|$, the size of the R-interval $[\ell_t, r_t]$.

1. $|x_t| = 1$. Suppose there is a single R-site on the line located at $\ell_t = r_t$. To determine the state of Config-$[\ell_0, r_0]$ in period $t+1$, it is sufficient to consider the successes and failures of sites BRB in period t, since all other B-sites have blue neighborhoods and do not change color in period $t+1$. Note that the lemma would be false only if BRB changes to RBR in period $t+1$. But this is an impossible event, because in order for the central R in BRB to switch to B, it must have failed in period t. Then rule L_1 prescribes that its two neighboring blues should not switch to R in period $t+1$.

2. $|x_t| \geq 2$. There are at least two R-sites in the interval $[\ell_t, r_t]$. We only need to consider the experimental outcomes of the two leftmost sites BR and the two rightmost sites RB in $BRR \cdots RRB$ at time t, where the cardinality of the central R-interval, $R \cdots R$, equals $|x_t| - 2 \geq 0$. Clearly, the lemma is valid as long as in period $t+1$,

   (a) the two leftmost sites BR in $BRR \cdots RRB$ do not change to RB, and

   (b) the two rightmost sites RB in $BRR \cdots RRB$ do not change to BR.

   Since (b) is a mirror image of (a), we only need to prove case (a). Since B in BR has a blue left-neighbour and R in BR has a red right-neighbour, by rule L_1, B will learn from R when his technology is a failure, and vice versa. Thus, if (a) were false, both B and R must have failed in period t. Then by rule L_1, neither B nor R would have switched his technology.

We thus conclude in period $t+1$ Config-$[\ell_0, r_0]$ must be either in state $\mathcal{B}$ or in a state of the form $[\ell_{t+1}, r_{t+1}]$, for some $\ell_{t+1} \leq r_{t+1}$. ■

¿From Lemma 1, we can treat Config-$[\ell_0, r_0]$ as a Markov chain $\{X_t, t = 0, 1, 2, \ldots \big| X_0 = [\ell_0, r_0]\}$, $\ell_0 \leq r_0$, where X_t, $t = 0, 1, \ldots$, assume values in the state space

$$\mathcal{S} = \{\mathcal{B} \cup \mathcal{R} \cup [\ell, r] : \ell \leq r, \ \ell, r = 0, \pm 1, \pm 2, \ldots\}. \tag{1}$$

Associated with Markov chain $\{X_t, t = 0, 1, 2, \ldots \big| X_0 = [\ell_0, r_0]\}$ is its *cardinality process* , $\{|X_t|, t = 0, 1, \ldots \big| |X_0| = r_0 - \ell_0 + 1\}$, where $|X_t|$ is the number of R-sites in $[L_t, R_t]$ in period t:

$$|X_t| = \begin{cases} 0 & \text{if } X_t = \mathcal{B}, \\ R_t - L_t + 1 & \text{if } X_t = [L_t, R_t]. \end{cases} \tag{2}$$

Evidently, the cardinality process$\{|X_t|, t = 0, 1, \ldots \big| |X_0| = r_0 - \ell_0 + 1\}$ is also a Markov chain defined on the right half line Z_+. Note that $|X_t| = 0$ means that Markov chain $\{X_t, t = 0, 1, \ldots\}$ is absorbed by state $\mathcal{B}$ prior to time t.

Lemma 2 *Let $\{|X_t|, t = 0, 1, \ldots \big| |X_0| = i\}$ be the Markov chain defined by (2), with initial state $|X_0| = i$. For $i = 2, 3, \ldots$, let T_i be the stopping time of the event*

$$T_i = \min\{t : 0 \leq t \leq \infty, \ |X_t| \in \{0, 1\} \big| |X_0| = i\}, \tag{3}$$

where $T_i = \infty$ if the above event never occurs. Then, for $i = 2, 3, \ldots$,

$$P(|X_{Ti}| = +\infty \ \big| |X_0| = i) := \gamma_i > 0, \tag{4}$$

$$P(|X_{Ti}| \in \{0, 1\} \big| |X_0| = i) = 1 - \gamma_i. \tag{5}$$

Proof. If $|X_t| = 1$, then $|X_{t+1}|$ depends only on the outcomes of sites BRB in period t. Let $|X_{t+1}| = |X_t| + Y_t^1$, where Y_t^1 is the net gain of R-sites in period $t+1$, given $|X_t| = 1$. It is easily seen that Y_t^1 follows the distribution

$$\begin{aligned} P(Y_t^1 = -1) &= (1 - p_R)(1 - (1 - p_B)^2), \\ P(Y_t^1 = 0) &= p_R p_B^2 + (1 - p_R)(1 - p_B)^2, \\ P(Y_t^1 = 1) &= 2 p_R p_B (1 - p_B), \\ P(Y_t^1 = 2) &= p_R (1 - p_B)^2. \end{aligned}$$

The mean of Y_t^1 works out to be

$$E[Y_t^1] = 2(p_R - p_B) + p_B^2(1 - p_R). \tag{6}$$

Now consider $|X_t| \geq 2$. In Lemma 1 (2), we have shown that $|X_{t+1}|$ depends on the successes and failures of the two leftmost sites and the two rightmost sites of $BR\,R \cdots R\,RB$, since all other sites do not switch technologies in period $t+1$. Let $|X_{t+1}| = |X_t| + Y_t^L + Y_t^R$, where Y_t^L and Y_t^R are the net gains of R-sites in period $t+1$ from sites BR and RB in $BRR \cdots RRB$, respectively. Since sites BR only learn from each other and similarly sites RB, random variables Y_t^L and Y_t^R, given $|X_t| \geq 2$, are independent and identically distributed with the distribution

$$\begin{array}{lll} P(Y_t^L = -1) & = P(Y_t^R = -1) & = (1 - p_R)p_B, \\ P(Y_t^L = 0) & = P(Y_t^R = 0) & = p_R p_B + (1 - p_R)(1 - p_B), \\ P(Y_t^L = 1) & = P(Y_t^R = 1) & = p_R(1 - p_B), \end{array}$$

and the expectation

$$E[Y_t^L] = E[Y_t^R] = p_R - p_B > 0. \tag{7}$$

Now consider (4). We have

$$\begin{aligned} P(|X_{T_i}| = +\infty \;\big|\; |X_0| = i) = {} & P(|X_{T_i}| = +\infty,\ T_i = \infty \;\big|\; |X_0| = i) \\ & + P(|X_{T_i}| = +\infty,\ T_i < \infty \;\big|\; |X_0| = i), \end{aligned} \tag{8}$$

where the stopping time T_i is defined in (3). First note that the second probability on the right hand side (RHS) of (8) must be zero, since a configuration starting with a finite R-interval and with the maximum net gain less than or equal to 2 in each period cannot be absorbed by $\mathcal{R}$ in a finite time $T_i < \infty$. Thus (8) reduces to

$$P(|X_{T_i}| = +\infty \;\big||X_0| = i) = P(|X_{T_i}| = +\infty,\ T_i = \infty \;\big||X_0| = i). \tag{9}$$

Equation (9) is the probability that the Markov chain $\{|X_t|, t = 0, 1, \ldots T_i \big| \, |X_0| = i\}$ is absorbed by $+\infty$ at time $T_i = \infty$. To prove this probability is strictly positive, observe that the Markov chain is equivalent to the following random walk on Z_+: The random walk starts in state $i \in \{2, 3, \ldots\}$ and has i.i.d drifts $Y_t^L + Y_t^R$ with a positive mean $E[Y_t^L + Y_t^R] = 2(p_R - p_B) > 0$. The random walk ends as soon as it reaches either state 0 or state 1. It is well-known (see, for example, Feller (1971)) that such a random walk either drifts to $+\infty$ in equilibrium or is absorbed by states $\{0,1\}$, both with strictly positive probabilities, say γ_i and $1-\gamma_i$, respectively. This proves (8) and henceforth (4). Similarly, (5) can be written as

$$\begin{aligned} P(|X_{T_i}| \in \{0,1\} \,\big||X_0| = i) &= P(|X_{T_i}| \in \{0,1\},\ T_i = \infty \,\big||X_0| = i) \\ &\quad + P(|X_{T_i}| \in \{0,1\},\ T_i < \infty \,\big||X_0| = i) \\ &= P(|X_{T_i}| \in \{0,1\},\ T_i < \infty \,\big||X_0| = i), \qquad (10) \end{aligned}$$

where $P(|X_{T_i}| \in \{0,1\},\ T_i = \infty \,\big||X_0| = i) = 0$ because $\{T_i = \infty\}$ means that the chain never visits states $\{0,1\}$ and then the random walk result states that the chain at $\{T_i = \infty\}$ must drift to $+\infty$. The last equation of (10) corresponds to the probability that the aforementioned random walk is absorbed by states $\{0,1\}$, which equals $1-\gamma_i$ as we have shown. This proves (5). ■

The next proposition states that given $X_0 = [\ell_0, \ell_0+i-1]$, $i \geq 2$, the Markov chain $\{X_t, t=0,1,\ldots \big|\ X_0 = [\ell_0,\ell_0+i-1]\}$ at stopping time T_i is in state $\mathcal{R}$ with probability $\gamma_i > 0$ and in the set of states $\{\mathcal{B}, [\ell,\ell],\ \ell = 0, \pm 1, \pm 2, \ldots\}$ with probability $1-\gamma_i$.

Proposition 3 *Let* $\{X_t, t = 0,1,\ldots \big| X_0 = [\ell_0, \ell_0+i-1]\}$ *be the Markov chain associated with Config-*$[\ell_0,\ell_0+i-1]$*, with state space* $\mathcal{S}$ *given in (1). Then for* $i = 2,3,\ldots$,

$$\begin{aligned} P(X_{T_i} = \mathcal{R} \big| X_0 = [\ell_0,\ell_0+i-1]) &= \gamma_i, \qquad (11) \\ P(X_{T_i} \in \{\mathcal{B}, [\ell,\ell], \ell = 0, \pm 1, \ldots\} \big| X_0 = [\ell_0,\ell_0+i-1]) &= 1-\gamma_i. \qquad (12) \end{aligned}$$

Proof. First note that since the line extends to infinity in both directions, the probabilities given in (11) and (12) depend only on the cardinality of X_0 and are independent of the location of X_0. Therefore,

$$P(X_{T_i} = \mathcal{R} \big| X_0 = [\ell_0,\ell_0+i-1]) = P(X_{T_i} = \mathcal{R} \big|\ |X|_0 = i), \qquad (13)$$

and similarly in (11) we can replace $X_0 = [\ell_0, \ell_0+i-1]$ by $|X_0| = i$.

Now conditioning on $|X_{T_i}|$, we express (13) as

$$\begin{aligned} & P(X_{T_i} = \mathcal{R} \big|\ |X_0| = i) \\ &\quad = P(X_{T_i} = \mathcal{R} \big|\ |X_0| = i, |X_{T_i}| < \infty) P(|X_{T_i}| < \infty \ \big|\ |X_0| = i) \\ &\qquad + P(X_{T_i} = \mathcal{R} \big|\ |X_0| = i, |X_{T_i}| = \infty) P(|X_{T_i}| = \infty \ \big|\ |X_0| = i) \\ &\quad = \gamma_i P(X_{T_i} = \mathcal{R} \big|\ |X_0| = i, |X_{T_i}| = \infty), \qquad (14) \end{aligned}$$

where we have used (4), (10) and the fact that $\{X_{T_i} = \mathcal{R} \big| \ |X_0| = i, |X_{T_i}| < \infty\}$ is an impossible event and thus has a null probability. Therefore, to prove (11), we need to show

$$P(X_{T_i} = \mathcal{R} \big| \ |X_0| = i, |X_{T_i}| = \infty) = 1. \tag{15}$$

In other words, if the cardinality of a R-interval is $+\infty$, then the boundaries of the R-interval must extend to infinity in both directions. Toward this end, note from (8) that

$$\{|X_{T_i}| = \infty \ \big| \ |X_0| = i\} \Longleftrightarrow \{|X_{T_i}| = \infty, T_i = \infty \ \big| \ |X_0| = i\}.$$

If the above event occurs, the process $\{|X_t|, t = 0, 1, \ldots, T_i = \infty \ \big| \ |X_0| = i\}$ never visits states 0 and 1 and thus there are at least two R-sites in the interval $X_t = [L_t, R_t]$ for $t = 0, 1, \ldots, T_i = \infty$. As we argued previously, in this case $\{Y_t^L,\ t = 0, 1, \ldots, T_i = \infty\}$ is a sequence of i.i.d random variables, independent of the sequence $\{Y_t^R,\ t = 0, 1, \ldots, T_i = \infty\}$. Thus, $\{L_t, t = 0, \ldots, T_i \big| \ |X_0| = i, |X_{T_i}| = \infty\}$ can be viewed as a random walk on $(-\infty, +\infty)$ with initial state ℓ_0 and i.i.d random drifts $-Y_t^L$, $t = 0, 1, \ldots, T_i = \infty$. Since the expected drift $-E[Y_t^L]$ is negative, it is known (again, see Feller 1971) that such a random walk drifts to $-\infty$ as $t \to \infty$, with probability 1. Thus

$$P(L_{T_i} = -\infty \ \big| |X_0| = i, |X_{T_i}| = \infty) = 1. \tag{16}$$

Similarly, $\{R_t, t = 0, \ldots, T_i \big| \ |X_0| = i, |X_{T_i}| = \infty)$ is a random walk on $(-\infty, +\infty)$ with initial state r_0 and i.i.d random drifts Y_t^R, $t = 0, 1, \ldots, T_i = \infty$. Since $E[Y_t^R] > 0$, the random walk drifts to $+\infty$ with probability 1 as $t \to \infty$. Thus

$$P(R_{T_i} = +\infty \ \big| \ |X_0| = i, |X_{T_i}| = \infty) = 1. \tag{17}$$

Combining (16) and (17),

$$\begin{aligned} &P(X_{T_i} = \mathcal{R} \big| \ |X_0| = i, |X_{T_i}| = \infty) \\ &= P(L_{T_i} = -\infty,\ R_{T_i} = +\infty \big| \ |X_0| = i, |X_{T_i}| = \infty) = 1. \end{aligned} \tag{18}$$

This proves (15) and also (11). The proof of (12) is analogous and we omit the details. ■

2.2 Sequential Coupling

We assume that in the original configuration, each agent on the line is independently assigned $R(=1)$ with probability $0 < q_R < 1$ and $B(=0)$ with probability $1 - q_R$, at time 0. For convenience, we call the original process that starts with a randomly assigned technology at each site *Config-O*. It is clear that Config-O at time 0 has infinitely many R-intervals with their cardinalities at least 2.

The proof in this section proceeds as follows. From the original process (Config-O), which is denoted by $\tilde{\mathbf{Z}}_t, t = 0, 1, \ldots$, we choose an interval of all reds that has a cardinality of at least two at $t = 0$. We couple this with an auxiliary process that has only two reds at time 0 and is "covered" by the interval chosen from the original process, in the sense of having blues wherever the original process had blues. The first part of the proof demonstrates that we can find a coupling such that if this covering holds at time t,it holds for $t + 1$.To this end a process $\hat{\mathbf{Z}}_t$ is constructed such that the marginal distributions of $\hat{\mathbf{Z}}_t$ and $\tilde{\mathbf{Z}}_t$ are the same for all t, and the location of the chosen red interval is the same in both processes at $t = 0$. By appropriately choosing random variables that move together in the two coupled processes, the construction maintains the covering property for all t for the adjunct process that has an interval with only two reds at time 0. The existence of such a sample path inequality is equivalent to a distributional inequality in which the leftmost point of the original interval at time t is stochastically less than the leftmost point of the interval that started with only two reds and the rightmost point is stochastically greater. Thus the process starting with two reds serves as a stochastic lower bound for the evolution of the chosen red interval in the original process. We know the size-two interval either goes to infinity with positive probability or is absorbed in a size-1 or size-0 state with positive probability. If the latter happens, we restart with a new red interval of cardinality at least two in the original process and so on. [3]

Let $\tilde{\mathbf{Z}}_0 = \{\tilde{Z}_{0,i}, i = 0, \pm 1, \pm 2, \ldots\}$ be the state of Config-O at time zero, where $\tilde{Z}_{0,i}$, $i = 0, \pm 1, \pm 2, \ldots$, are i.i.d random variables with $P(\tilde{Z}_{0,i} = R) =$

[3] The coupling technique is a standard tool in probability theory; the general procedure is to construct for the process of interest $\widetilde{X}_t$ an auxiliary process X_t such that the marginal distribution of X_t is the same as that of $\widetilde{X}_t$, for all t. The two coupled processes move together using the realisation of the same random variable, and sample path inequalities on a particular realised trajectory are equivalent to distributional inequalities among the two processes, so that one can serve as a stochastic lower bound (for example) for the other.See the references given earlier for more details on the general technique.

q_R and $P(\tilde{Z}_{0,i} = B) = 1 - q_R$. Let $\{\tilde{\mathbf{Z}}_t, t = 0, 1, \ldots\}$ represent the process associated with Config-O, with initial state $\tilde{\mathbf{Z}}_0$. It is clear that $\{\tilde{\mathbf{Z}}_t, t = 0, 1, \ldots\}$ is a Markov chain. Let $\tilde{X}_0 = [\tilde{\ell}_0, \tilde{r}_0]$, $\tilde{r}_0 > \tilde{\ell}_0$ be one of the R-intervals (with cardinality ≥ 2) from Config-O at time 0. Denote the location of the R-interval in period t by $\tilde{X}_t = [\tilde{L}_t, \tilde{R}_t]$, where $\tilde{L}_t \leq \tilde{R}_t$ means that the interval still exists by t and $\tilde{L}_t > \tilde{R}_t$ means that the R-interval ceases to exist by t, $t = 0, 1, \ldots$.

Let $\{X_t, t = 0, 1, \ldots \big| X_0 = [\ell_0, \ell_0 + 1]\}$ be the Markov chain associated with *Config*-$[\ell_0, \ell_0 + 1]$, a configuration that starts with only two R-sites located at integer points $\{\ell_0, \ell_0 + 1\}$, respectively, at time 0, $\ell_0 = 0, \pm 1, \pm 2, \ldots$. Similar to (3), let T_2 be the first time that Config-$[\ell_0, \ell_0 + 1]$ visits states $\{\mathcal{B} \cup [\ell, \ell], \ell = 0, \pm 1, \pm 2, \ldots\}$:

$$
\begin{aligned}
&T_2 = \min\{t : 0 \leq t \leq \infty, X_t \in \{\mathcal{B}, [\ell, \ell], \ell = 0, \pm 1, \ldots\} \big| X_0 = [\ell_0, \ell_0 + 1]\} \\
&=_{st} \min\{t : 0 \leq t \leq \infty, |X_t| \in \{0, 1\}\} \big| \, |X_0| = 2\}, \qquad (19)
\end{aligned}
$$

where $X =_{st} Y$ means that X and Y are equal in distribution.

Lemma 4 *Let $\tilde{X}_0 = [\tilde{\ell}_0, \tilde{r}_0]$, $\tilde{r}_0 > \tilde{\ell}_0$, be a R-interval from Config-O at time 0 and let $\tilde{X}_t = [\tilde{L}_t, \tilde{R}_t]$ be the location of the R-interval at time t, where we take the convention that $\tilde{L}_t > \tilde{R}_t$ indicates that the R-interval no longer exists at time t. Let Config-$[\ell_0, \ell_0 + 1]$ satisfy, in period 0,*

$$X_0 = [\ell_0, \ell_0 + 1] \subseteq [\tilde{\ell}_0, \tilde{r}_0] = \tilde{X}_0. \qquad (20)$$

Denote the state of Config-$[\ell_0, \ell_0 + 1]$ in period t by $X_t = [L_t, R_t]$, $|X_t| \geq 2$, $t = 0, 1, \ldots, T_2$, where T_2 is the stopping time defined in (19). Then, for any $t = 0, 1, \ldots, T_2$,

$$
\begin{aligned}
\tilde{L}_t &\leq_{st} \quad L_t, \qquad (21) \\
\tilde{R}_t &\geq_{st} \quad R_t. \qquad (22)
\end{aligned}
$$

Proof. We shall construct a configuration, call it Config-Ô, governed by the Markov chain $\{\hat{\mathbf{Z}}_t, t = 0, 1, \ldots, T_2 \big| \hat{\mathbf{Z}}_0 = \tilde{\mathbf{Z}}_0\}$, such that it obeys the same probability law as $\{\tilde{\mathbf{Z}}_t, t = 0, 1, \ldots, T_2\}$:

$$
\begin{aligned}
\hat{\mathbf{Z}}_0 &= \tilde{\mathbf{Z}}_0 \qquad (23) \\
\hat{\mathbf{Z}}_{t+1} \big| \hat{\mathbf{Z}}_t &=_{st} \tilde{\mathbf{Z}}_{t+1} \big| \tilde{\mathbf{Z}}_t, \quad t = 0, 1, \ldots, T_2, \qquad (24)
\end{aligned}
$$

but, for $t = 0, 1, \ldots, T_2$,

$$\hat{L}_t =_{st} \tilde{L}_t \tag{25}$$

and

$$\hat{L}_t \leq L_t \quad \text{with probability 1.} \tag{26}$$

Similarly,

$$\hat{R}_t =_{st} \tilde{R}_t \tag{27}$$

and

$$\hat{R}_t \geq R_t \quad \text{with probability 1,} \tag{28}$$

where $\hat{X}_t = [\hat{L}_t, \hat{R}_t]$ is the location of the R interval, with initial location $\hat{X}_0 = \tilde{X}_0 = [\widetilde{\ell_0}, \widetilde{r_0}]$, at time t in Config-Ô. Then, from the result of stochastic ordering (see, e.g., Proposition 9.2.2, Ross 1996), (25) -(28) imply (21) and (22), respectively.

Our sample path construction proceeds on t. Equations (23)-(28) hold trivially for $t = 0$, due to (20). Next we show (23)-(28) hold for $t + 1$, based on the hypotheses they hold for t. To construct $\hat{L}_{t+1}$ and $\hat{R}_{t+1}$, we couple sites BR located at sites $\{\hat{L}_t - 1, \hat{L}_t\}$ in Config-Ô with sites BR located $\{L_t - 1, L_t\}$ in Config-$[\ell_0, \ell_0 + 1]$ so that they yield the identical outcomes at time t. Similarly, we couple sites RB located at sites $\{\hat{R}_t, \hat{R}_t + 1\}$ in Config-Ô with sites RB located $\{R_t, R_t + 1\}$ in Config-$[\ell_0, \ell_0 + 1]$ so that they yield the identical outcomes at time t. Note that the left and right boundary couplings do not interfere with each other because for $0 \leq t < T_2$, the cardinalities of the R-intervals in both configurations are at least two. Let any other site in Config-Ô, except for the two left-boundary sites and the two right-boundary sites identified above, have its own realization of experimental outcome at time t, independent of everything else.

Due to the Markov property, the probability law governing $\hat{\mathbf{Z}}_{t+1}$ ($\tilde{\mathbf{Z}}_{t+1}$) depends only on $\hat{\mathbf{Z}}_t$ ($\tilde{\mathbf{Z}}_t$) and their experimental outcomes at time t. From our hypothesis, (23) holds for $t - 1$:

$$\hat{\mathbf{Z}}_t \big| \hat{\mathbf{Z}}_{t-1} =_{st} \tilde{\mathbf{Z}}_t \big| \tilde{\mathbf{Z}}_{t-1}.$$

In addition, the couplings prescribed above ensure that the experimental outcomes of $\hat{\mathbf{Z}}_t$ are stochastically identical to that of $\tilde{\mathbf{Z}}_t$. Therefore,

$$\hat{\mathbf{Z}}_{t+1} \big| \hat{\mathbf{Z}}_t =_{st} \tilde{\mathbf{Z}}_{t+1} \big| \tilde{\mathbf{Z}}_t, \tag{29}$$

which proves (23) for $t + 1$.

Next we prove (27)-(28). Due to symmetry, (25)-(26) can be shown analogously. Toward this end, let $\hat{Y}_t^R$, $\tilde{Y}_t^R$ and Y_t^R be the net gains of R-sites in Config-Ô, Config-O and Config-$[\ell_0, \ell_0+1]$ in period $t+1$ from the right boundaries of their respective R-intervals. Since $\hat{Y}_t^R$ depends only on the outcomes of $\{\hat{Z}_{t,i}, i \geq \hat{R}_t\}$, and $\tilde{Y}_t^R$ depends only on the outcomes of $\{\tilde{Z}_{t,i}, i \geq \tilde{R}_t\}$, our hypothesis (23)-(28) and the prescribed coupling imply that

$$\hat{Y}_t^R =_{st} \tilde{Y}_t^R. \tag{30}$$

In addition, $\hat{R}_t$ is independent of $\hat{Y}_t^R$, since $\hat{Y}_t^R$ depends only on the output of the site $\hat{R}_t$ (that is, $\hat{Z}_{t,\hat{R}_t}$) but not the location $\hat{R}_t$ itself. Similarly, $\tilde{Y}_t^R$ is independent of $\tilde{R}_t$. Then, from (27) and (30),

$$\hat{R}_{t+1} = \hat{R}_t + \hat{Y}_t^R =_{st} \tilde{R}_t + \tilde{Y}_t^R = \tilde{R}_{t+1}, \tag{31}$$

which proves (27) for $t+1$. Finally, to prove (28) for $t+1$, recall that our coupling requires that the sites RB located at $\{\hat{R}_t, \hat{R}_t + 1\}$ in Config-Ô in period t have the identical outcomes as the sites RB located in $\{R_t, R_t+1\}$ in Config-$[\ell_0, \ell_0+1]$ in period t. Therefore, if at time t,

- The sites RB in Config-$[\ell_0, \ell_0+1]$ were (f, s), where f stands for failure and s success, then RB in both configurations become BB at time $t+1$ and hence $\hat{Y}_t^R = Y_t^R = -1$;
- RB in Config-$[\ell_0, \ell_0+1]$ were (s, s) or (f, f), then RB in Config-$[\ell_0, \ell_0+1]$ are still RB, but RB in Config-$\hat{O}$ become either RB or RR at time $t+1$, depending on whether there is a R on the right of the B and the R succeeds. Therefore, $\hat{Y}_t^R \geq Y_t^R = 0$;
- RB in Config-$[\ell_0, \ell_0+1]$ were (s, f), then RB in both configurations become RR at time $t+1$ and we have $\hat{Y}_t^R \geq Y_t^R = 1$.

Those events imply

$$\hat{Y}_t^R \geq Y_t^R \quad \text{with probability 1.} \tag{32}$$

Then (32), together with (28), further implies

$$\begin{aligned}\hat{R}_{t+1} &= \hat{R}_t + \hat{Y}_t^R \\ &\geq R_t + Y_t^R = R_{t+1}, \quad \text{with probability 1,}\end{aligned} \tag{33}$$

this establishes (28) for $t+1$, and also our induction proof. ■

The next theorem shows that if $p_R > p_B$, then $\mathcal{R}$ prevails with probability 1 in Config-O, as $t \to \infty$.

Theorem 5 *Under rule L_1, Config-O will be absorbed by $\mathcal{R}$ as $t \to \infty$, with probability 1.*

Proof. We establish the theorem by repeatedly coupling Config-Ô^j with Config-$[\ell_0^j, \ell_0^j+1]$, $j = 1, 2, \ldots$, until Config-$[\ell_0^j, \ell_0^j+1]$ is absorbed by $\mathcal{R}$ for some j.

Specifically, at time zero we arbitrarily select a R-interval, say $\tilde{X}_0 = [\tilde{\ell}_0, \tilde{r}_0]$, $\tilde{r}_0 > \tilde{\ell}_0$, from Config-O. Accordingly, we select Config-$[\ell_0^1, \ell_0^1+1]$ such that $[\tilde{\ell}_0, \tilde{r}_0] \supseteq [\ell_0^1, \ell_0^1]$. Let T_2^1 be the stopping time defined in (19). As shown in Lemma 4, we can construct Config-Ô^1, $\{\hat{\mathbf{Z}}_t^1, t = 0, 1, \ldots, T_2^1 | \hat{\mathbf{Z}}_0^1 = \tilde{\mathbf{Z}}_0\}$, where $\hat{\mathbf{Z}}_t^1$ is the state of Config-Ô^1 at time t, such that

$$\{\tilde{\mathbf{Z}}_t, t = 0, 1, \ldots, T_2^1\}$$
$$=_{st} \{\hat{\mathbf{Z}}_t^1, t = 0, 1, \ldots, T_2^1 | \hat{\mathbf{Z}}_0^1 = \tilde{\mathbf{Z}}_0\},$$

and for $t = 0, 1, \ldots, T_2^1$,

$$[\hat{L}_t^1, \hat{R}_t^1] \supseteq [L_t^1, R_t^1], \quad \text{with probability 1},$$

where $[\hat{L}_t^1, \hat{R}_t^1]$ and $[L_t^1, R_t^1]$ are the locations of the R-intervals at time t in Config-Ô^1 and Config-$[\ell_0^1, \ell_0^1+1]$, respectively. If $T_2^1 < \infty$, that is, if the R-interval in Config-$[\ell_0^1, \ell_0^1+1]$ either disappears or is reduced to a single R-site at a finite time T_2^1, then we select another R-interval from Config-O at time T_2^1, say $\tilde{X}_{T_2^1} = [\tilde{L}_{T_2^1}, \tilde{R}_{T_2^1}]$, $\tilde{R}_{T_2^1} > \tilde{L}_{T_2^1}$. Such a selection is possible because there are infinitely many R-intervals with their cardinalities at least 2 at time 0 and those R-intervals cannot all disappear or become a single R-site in a finite time $T_2^1 < \infty$. Correspondingly, we select Config-$[L_0^2, L_0^2+1]$ so that $[\tilde{L}_{T_2^1}, \tilde{R}_{T_2^1}] \supseteq [L_0^2, L_0^2+1]$. From Lemma 4 again, we can construct Config-Ô^2, $\{\hat{\mathbf{Z}}_t^2, t = T_2^1, \ldots, T_2^1 + T_2^2 | \hat{\mathbf{Z}}_{T_2^1} = \tilde{\mathbf{Z}}_{T_2^1}\}$, where $\hat{\mathbf{Z}}_t^2$ is the state of Config-Ô^2 at time $t = T_2^1, \ldots, T_2^1 + T_2^2$, such that

$$\{\tilde{\mathbf{Z}}_t, t = T_2^1, T_2^1+1, \ldots, T_2^1 + T_2^2\}$$
$$=_{st} \{\hat{\mathbf{Z}}_t^2, t = T_2^1, T_2^1+1, \ldots, T_2^1 + T_2^2 | \hat{\mathbf{Z}}_{T_2^1} = \tilde{\mathbf{Z}}_{T_2^1}\},$$

and for $t = T_2^1, \ldots, T_2^1 + T_2^2$,

$$[\hat{L}_t^2, \hat{R}_t^2] \supseteq [L_t^2, R_t^2], \quad \text{with probability } 1,$$

where $[\hat{L}_t^2, \hat{R}_t^2]$ and $[L_t^2, R_t^2]$ are the R-intervals at time t in Config-Ô^2 and Config-$[L_0^2, L_0^2+1]$, respectively.

Let $\mathcal{T}^k = \sum_{j=1}^k T_2^j$, where T_2^j is the stopping time of Config-$[L_0^j, L_0^j+1]$, $j = 1, 2, \ldots$. We repeat the above coupling process until time $\mathcal{T}^J$, where

$$J = \min\{j : 1 \leq j < \infty, T_2^j = \infty\}.$$

Since $T_2^j < \infty$ with probability $1 - \gamma_2$ and $T_2^j = \infty$ with probability γ_2, $j = 1, 2, \ldots$, J follows a geometric distribution with $\gamma_2 > 0$ and is finite with probability 1. Thus, at time $\mathcal{T}^J$, Config-$[L_0^J, L_0^J+1]$ will be absorbed by $\mathcal{R}$. Finally, because

$$\begin{aligned}&\{\tilde{\mathbf{Z}}_t, t = \mathcal{T}^{j-1}, \ldots, \mathcal{T}^j\} \\ &\qquad =_{st} \{\hat{\mathbf{Z}}_t^j, t = \mathcal{T}^{j-1}, \ldots, \mathcal{T}^j \big| \hat{\mathbf{Z}}_{\mathcal{T}_{j-1}} = \tilde{\mathbf{Z}}_{\mathcal{T}_{j-1}}\}, \quad j = 1, 2, \ldots, J,\end{aligned}$$

and for $j = 1, 2, \ldots, J$,

$$[\hat{L}_t^j, \hat{R}_t^j] \supseteq [L_t^j, R_t^j], \quad \mathcal{T}^{j-1} \leq t \leq \mathcal{T}^j, \text{with probability } 1,$$

we conclude

$$\begin{aligned}P([\tilde{L}_{\mathcal{T}^J}, \tilde{R}_{\mathcal{T}^J}] = \mathcal{R}) &= P([\hat{L}_{\mathcal{T}^J}^J, \hat{L}_{\mathcal{T}^J}^J] = \mathcal{R}) \\ &\geq P([L_{\mathcal{T}^J}^J, R_{\mathcal{T}^J}^J] = \mathcal{R}) = 1.\end{aligned}$$

This completes the proof of Theorem 5. ■

Note that we have assumed randomly chosen initial conditions in this proof. It is clear that the result does not hold if we start from the *All Blue* configuration. However, any initial configuration that has an infinite number of *Reds* will give the same result. This can be argued as follows. If initially there are finitely many R-intervals in Config-O, then there must be at least one R-interval with infinitely many reds. Clearly, this R-interval always has infinite reds for finite t and our sequential coupling approach is applicable. On the other hand, if initially there are infinitely many R-intervals in Config-O, then it is easily seen that this initial configuration can generate infinite R-interval of length at least 2 in the next period and we will start our coupling from the next period. As a specific example, consider the initial configuration

$$.....RBRBR........$$

Here there is no initial red interval of length at least 2 so we cannot begin the sequential coupling immediately. However, it is clear that, with probability 1, there will be infinitely many red intervals of length at least 2 from the next period onwards. We begin the first coupling at this time and our sequential coupling approach can still apply.

3 Technology Diffusion under Rule L_2

Under learning rule L_2, agent i performs his experiment, observes what technology $i-1, i+1$ and, of course, i himself are using, and also whether these neighbouring agents obtained successes or failures. If i is using R, and the proportion of successes of agents in his neighbourhood using B is *strictly* greater than the proportion of successes of agents using R, i switches to B in the next period and similarly for any agent using B. The switching probabilities of the central agent i under rule L_2 are as follows:

1. BBB or RRR: No switch;
2. BBR or RBB: Switch with probability $(1-p_B^2)p_R$;
3. RRB or BRR: Switch with probability $(1-p_R^2)p_B$;
4. BRB: Switch with probability $(1-p_R)(1-(1-p_B)^2)$;
5. RBR: Switch with probability $(1-p_B)(1-(1-p_R)^2)$.

Note that only under cases (2) and (3) above are the switching probabilities different from their counterparts under rule L_1. Observe also that agent i is more likely to switch to the alternative technology under rule L_2 than under rule L_1. (Under rule L_1, an agent never switched if he obtained a success.)

Consider Config-$[\ell_0, r_0]$ described in Section 2.1. Recall that under rule L_1, Config-$[\ell_0, r_0]$ always consists of a single R-interval as time evolves until absorption by either $\mathcal{R}$ or $\mathcal{B}$. Unfortunately, this property no longer holds under rule L_2. To see this, consider a R-interval with at least four reds so that the switches at the left and right boundary sites are independent. Let us consider $RRBB$ located at the neighbourhood of the right boundary of the R-interval. If the outcomes of $RRBB$ are (f, s, s, f), then in the next period those sites will change to $RBRB$, resulting in a red "hole" between two blues.

Another property of rule L_1, which is essential to our sequential coupling approach in Section 2, is that if the R-interval $X_t = [L_t, R_t]$ in Config-$[\ell_0, r_0]$ is "covered" by the R-interval $\hat{X}_t = [\hat{L}_t, \hat{R}_t]$ in Config-$\hat{O}$ at time t, then we are able to construct a coupling such that X_{t+1} is again "covered" by $\hat{X}_{t+1}$ at time $t+1$, with probability 1 (see Lemma 4). This property is invalid under rule L_2. For example, suppose $R_t = \hat{R}_t$, and the four sites located at $[R_t - 1, R_t, R_t + 1, R_t + 2] = [\hat{R}_t - 1, \hat{R}_t, \hat{R}_t + 1, \hat{R}_t + 2]$ in Config-$[\ell_0, r_0]$ and Config-$\hat{O}$ at time t are, respectively, $RRBB$ and $RRBR$. Suppose the outcome of those four sites in both configurations are (s, s, s, f). Then under L_2, $RRBB$ in Config-$[\ell_0, r_0]$ will change to $RRRB$ and $RRBR$ in Config-$\hat{O}$ will change to $RRBB$, and X_{t+1} is no longer covered by $\hat{X}_{t+1}$ at the right boundary at time $t+1$.

To overcome the above difficulties, we modify rule L_2 for Config-$[\ell_0, r_0]$ as follows:

M1. If the outcomes of $RRBB$ ($BBRR$) located at the right (left) boundary of X_t in Config-$[\ell_0, r_0]$ are (f, s, s, f), which occur with probability $(1 - p_R)p_R p_B(1 - p_B)$, then we let $RRBB$ ($BBRR$) become $RBBB$ ($BBBR$) in the next period. In other words, whenever a R-interval splits at the boundary of the interval we will change the rightmost (leftmost) red site to blue.

M2. If the outcomes of $RRBB$ ($BBRR$) located at the right (left) boundary of X_t in Config-$[\ell_0, r_0]$ are (s, s, s, f) $\big((f, s, s, s)\big)$, which occur with probability $p_R^2 p_B(1 - p_B)$, then we let $RRBB$ ($BBRR$) retain their states $RRBB$ ($BBRR$) in the next period.

It is worth mentioning that we only modify rule L_2 for Config-$[\ell_0, r_0]$. Rule L_2 is still enforced in the original configuration, Config-O.

M1 ensures that a configuration starting with a single red interval always consists of a single red interval until its absorbtion by $\mathcal{B}$, if that event ever occurs. As such, we can again treat Config-$[\ell_0, r_0]$ under the modified L_2 as a Markov chain $\{X_t, t = 0, 1, 2, \ldots \,\big|\, X_0 = [\ell_0, r_0]\}$, $r_0 \geq \ell_0$, where $X_t = [L_t, R_t]$

is the location of the R-interval at time t, $t = 0, 1, \ldots$. Let $\{|X_t|, t \geq 0 \| |X_0| = r_0 - \ell_0\}$ be the cardinality process of $\{X_t, t = 0, 1, 2, \ldots \big| X_0 = [\ell_0, r_0]\}$, with $|X_t|$ defined by (2). Also redefine T_i as the stopping time of the event

$$T_i = \min\{t : 0 \leq t \leq \infty,\ |X_t| < 4 \big| |X_0| = i \geq 4\}, \tag{34}$$

where $T_i = \infty$ if the above event never occurs.

The next lemma can be considered as an extension of Lemma 2 and Proposition 3.

Lemma 6 *Let $\{X_t, t = 0, 1, \ldots \mid X_0 = [\ell_0, \ell_0 + i - 1]\}$ be the Markov chain associated with Config-$[\ell_0, \ell_0 + i - 1]$ under the modified rule L_2.*

1. *If $\frac{p_R}{1-p_R^2} > \frac{p_B}{1-p_B}$, then for $i \geq 4$,*

$$P(|X_{T_i}| = +\infty \ \big| |X_0| = i) := \gamma_i > 0, \tag{35}$$

$$P(|X_{T_i}| < 4 \big| |X_0| = i) = 1 - \gamma_i. \tag{36}$$

2. *If $\frac{p_R}{1-p_R^2} > \frac{p_B}{1-p_B}$, then for $i \geq 4$,*

$$P(X_{Ti} = \mathcal{R} \big| X_0 = [\ell_0, \ell_0 + i - 1]) = \gamma_i, \tag{37}$$

$$P(X_{Ti} \in \{[\ell, \ell+1], [\ell, \ell+2], \ell = 0, \pm 1, \ldots\} \big| X_0 = [\ell_0, \ell_0+i-1]) = 1-\gamma_i. \tag{38}$$

Proof. We only prove (1), the proof of (2) is similar to that of Proposition 3.

Under the modified L_2, $|X_{t+1}|$ depends on the successes and failures of the four leftmost sites and the four rightmost sites of $BBRR\ R \cdots R\ RRBB$. Again let $|X_{t+1}| = |X_t| + Y_t^L + Y_t^R$, where Y_t^L and Y_t^R are the net gains of R-sites in period $t+1$ from sites $BBRR$ and $RRBB$ in $BBRR\ R \cdots R\ RRBB$, respectively. Because $|X_t| \geq 4$, Y_t^L and Y_t^R are independent and identically distributed and we only need to consider Y_t^R. Examining the proof of Lemma 2, it is sufficient to show that under the condition $\frac{p_R}{1-p_R^2} > \frac{p_B}{1-p_B}$, $E[Y_t^R] > 0$. Under the modified L_2, Y_t^R has the distribution

$$\begin{aligned} P(Y_t^R = -1) &= (1 - p_R)p_B + (1 - p_R)p_R p_B^2 + (1 - p_R)p_R p_B(1 - p_B), \\ P(Y_t^R = 1) &= p_R(1 - p_B), \\ P(Y_t^R = 0) &= (1 - p_R)(1 - p_B) + p_r^2 p_B^2 + p_R^2 p_B(1 - p_B), \end{aligned} \tag{39}$$

where the first equality holds because the event $Y_t^R = -1$ means that the outcomes of RB in $RRBB$ are (f, s), or the outcomes of $RRBB$ are (f, s, s, s), or the outcomes of $RRBB$ are (f, s, s, f), where the last case is due to M1. The second equality holds because the event $Y_t^R = 1$ means that the outcomes of RB in $RRBB$ are (s, f). The third equality holds because the event $Y_t^R = 0$ means that the outcomes of RB in $RRBB$ are (f, f), or the outcomes of $BBRR$ are (s, s, s, s), or the outcomes of $RRBB$ are (s, s, s, f), where the last case is the result of M2. We then work out the expectation as

$$E[Y_t^R] = p_R - p_B - p_R p_B + p_R^2 p_B \tag{40}$$

Thus the above expectation is positive as long as $\frac{p_R}{1-p_R^2} > \frac{p_B}{1-p_B}$. ■

Next consider the original configuration, Config-O, under rule L_2, where at time 0 each agent on the line is independently assigned R with probability $0 < q_R < 1$ and B with probability $1 - q_R$. Let $\{\tilde{\mathbf{Z}}_t, t = 0, 1, \ldots\}$ be similarly defined as in Section 2 but under rule L_2. Let $\tilde{X}_0 = [\tilde{\ell}_0, \tilde{r}_0]$, $\tilde{r}_0 - \tilde{\ell}_0 \geq 4$, be one of the R-intervals from Config-O at time 0. As time evolves, the location of the R-interval at time $t + 1 \geq 1$ is given by

$$[\tilde{L}_{t+1}, \tilde{R}_{t+1}] = [\tilde{L}_t - \tilde{Y}_t^L, \tilde{R}_t + \tilde{Y}_t^R], \quad t = 0, 1, \ldots, \tag{41}$$

where $\tilde{Y}_t^R$ ($\tilde{Y}_t^L$) is the net gain of the R-sites from the right (left) boundary of the interval, and it is understood that if RB (BR) in $BRR \cdots RRB$ at time t becomes BR (RB) at time period $t + 1$ (i.e., the R-interval splits), then the R-interval loses one R-site from the right (left) boundary and hence $\tilde{Y}_t^R = -1$ ($\tilde{Y}_t^L = -1$). Let $\{X_t, t = 0, 1, \ldots | X_0 = [\ell_0, \ell_0 + 3]\}$ be the Markov chain associated with Config-$[\ell_0, \ell_0 + 3]$, $\ell_0 = 0, \pm 1, \pm 2$, under the modified L_2. The next lemma extends Lemma 4.

Lemma 7 *Let Config-$[\ell_0, \ell_0 + 3]$ satisfy, in period 0,*

$$X_0 = [\ell_0, \ell_0 + 3] \subseteq [\tilde{\ell}_0, \tilde{r}_0] = \tilde{X}_0. \tag{42}$$

Let T_4 be defined by (34). If $\frac{p_R}{1-p_R^2} > \frac{p_B}{1-p_B}$, then for any $t = 0, 1, \ldots, T_4$,

$$\tilde{L}_t \leq_{st} \quad L_t, \tag{43}$$

$$\tilde{R}_t \geq_{st} \quad R_t. \tag{44}$$

Proof. Our approach to establish the result is analogous to that of Lemma 4, the only difference is that the coupling is constructed differently.

As in the proof of Lemma 4, we construct Config-Ô, governed by the Markov chain $\{\hat{\mathbf{Z}}_t, t = 0, 1, \ldots, T_4 \big| \hat{\mathbf{Z}}_0 = \tilde{\mathbf{Z}}_0\}$, such that it obeys the same probability law as $\{\tilde{\mathbf{Z}}_t, t = 0, 1, \ldots, T_4\}$:

$$\hat{\mathbf{Z}}_0 = \tilde{\mathbf{Z}}_0, \quad \hat{\mathbf{Z}}_{t+1} \big| \hat{\mathbf{Z}}_t =_{st} \tilde{\mathbf{Z}}_{t+1} \big| \tilde{\mathbf{Z}}_t, \quad t = 0, 1, \ldots, T_4, \tag{45}$$

but, for $t = 0, 1, \ldots, T_4$,

$$\tilde{L}_t =_{st} \hat{L}_t \text{ and } \hat{L}_t \leq L_t \text{ with probability 1}, \tag{46}$$

and

$$\tilde{R}_t =_{st} \hat{R}_t \text{ and } \hat{R}_t \geq R_t \text{ with probability 1}, \tag{47}$$

where $\hat{X}_t = [\hat{L}_t, \hat{R}_t]$ is the location of the R-interval at time t, with initial location $\hat{X}_0 = \tilde{X}_0$ in Config-Ô. We emphasize that as in Config-O, rule L_2 is implemented in Config-Ô.

¿From (42), (45)-(47) are trivially true for $t = 0$. Next we show (45)-(47) hold for $t + 1$, based on the hypothesis that they hold for t. Our coupling is constructed based on the configurations of the boundary sites of $\hat{X}_t$ and X_t, as follows. We couple the outcomes of RRB (BRR) at sites $\{R_t - 1, R_t, R_t + 1\}$ ($\{L_t - 1, L_t, L_t + 1\}$) in Config-$[\ell_0, \ell_0 + 3]$ with the outcomes of RRB (BRR) at sites $\{\widehat{R}_t - 1, \widehat{R}_t, \widehat{R}_t + 1\}$ ($\{\widehat{L}_t - 1, \widehat{L}_t, \widehat{L}_t + 1\}$) in Config-Ô so that they yield the identical outcomes at time t. In addition, if the color at site $\hat{R}_t + 2$ (site $\hat{L}_t - 2$) in Config-Ô is B, we couple its outcome with that of B at site $R_t + 2$ (site $L_t - 2$) in Config-$[\ell_0, \ell_0 + 3]$ so that their outcomes are identical; if the color at site $\hat{R}_t + 2$ (site $\hat{L}_t - 2$) in Config-Ô is R, we couple its outcome with that of B at site $R_t + 2$ (site $L_t - 2$) in Config-$[\ell_0, \ell_0 + 3]$ so that if B in Config-$[\ell_0, \ell_0 + 3]$ is a success then R in Config-Ô is also a success, at time t. This coupling is possible because $p_R > p_B$. Let any other site in Config-Ô or Config-$[\ell_0, \ell_0 + 3]$, except for the four left-boundary sites and the four right-boundary sites identified above, have its own outcome at time t, independent of everything else.

Next we show (45) and (47) hold for $t + 1$; the proof of (46) for $t + 1$ is analogous to that of (47). Let $\hat{Y}_t^R$, $\tilde{Y}_t^R$ and Y_t^R be the net gains of R-sites in Config-Ô, Config-O and Config-$[\ell_0, \ell_0 + 3]$ in period $t + 1$ from the right boundaries of their respective R-intervals. Following the similar arguments that lead to (29) and (31), we have

$$\hat{\mathbf{Z}}_{t+1} \big| \hat{\mathbf{Z}}_t =_{st} \tilde{\mathbf{Z}}_{t+1} \big| \tilde{\mathbf{Z}}_t, \tag{48}$$

and

$$\hat{R}_{t+1} = \hat{R}_t + \hat{Y}_t^R =_{st} \tilde{R}_t + \tilde{Y}_t^R = \tilde{R}_{t+1}, \tag{49}$$

which proves (45) and the first equation in (47) for $t+1$, respectively. Next we prove the second expression of (47) for $t+1$. For this, we consider several possible outcomes of the four coupled sites in the two configurations at time t.

- RB in $RRBB$ in Config-$[\ell_0, \ell_0+3]$ were (f, s), or $RRBB$ in Config-$[\ell_0, \ell_0+3]$ were (f, s, s, s) or (f, s, s, f). Then under the modified L_2, the sites $RRBB$ in Config-$[\ell_0, \ell_0+3]$ become $RBBB$. From the coupling described above, the four coupled sites $RRBB$ ($RRBR$) in Config-$\hat{O}$ become either $RBBB$, $RBRR$ or $RBRR$ ($RBBB$ or $RBBR$) at time $t+1$. Hence $\hat{Y}_t^R = Y_t^R = -1$.

- RB in $RRBB$ in Config-$[\ell_0, \ell_0+3]$ were (f, f), or $RRBB$ in Config-$[\ell_0, \ell_0+3]$ were (s, s, s, s) or (s, s, s, f). Then under the modified L_2, the sites $RRBB$ in Config-$[\ell_0, \ell_0+3]$ are still $RRBB$ at time $t+1$. From the coupling described above, the four coupled sites $RRBB$ ($RRBR$) in Config-$\hat{O}$ become either $RRBB$, $RRBR$, $RRRB$ or $RRRR$ ($RRBB$ or $RRBR$), at time $t+1$. Hence $\hat{Y}_t^R \geq Y_t^R = 0$;

- RB in $RRBB$ in Config-$[\ell_0, \ell_0+3]$ were (s, f), then $RRBB$ in Config-$[\ell_0, \ell_0+3]$ become $RRRB$ and $RRBB$ or $RRBR$ in Config-$\hat{O}$ become either $RRRB$ or $RRRR$, at time $t+1$. Therefore, $\hat{Y}_t^R \geq Y_t^R = 1$.

The above events imply that

$$\hat{Y}_t^R \geq Y_t^R, \quad \text{with probability } 1, \tag{50}$$

and it further implies

$$\begin{aligned} \hat{R}_{t+1} &= \hat{R}_t + \hat{Y}_t^R \\ &\geq R_t + Y_t^R = R_{t+1}, \quad \text{with probability } 1, \end{aligned} \tag{51}$$

where in the last inequality we used the hypothesis $\hat{R}_t \geq R_t$ with probability 1. This establishes the second part of (47) for $t+1$, and also our induction proof. ■

The next theorem states that if $\frac{p_R}{1-p_R^2} > \frac{p_B}{1-p_B}$, then $\mathcal{R}$ prevails with probability 1 in Config-O, as $t \to \infty$. Its proof is similar to that of Theorem 5 and we omit the details.

Theorem 8 *If $\frac{p_R}{1-p_R^2} > \frac{p_B}{1-p_B}$, then Config-O under L_2 will be absorbed by $\mathcal{R}$ as $t \to \infty$, with probability 1.*

Following the remark given at the end of Section 3, one sees that the result of this section is still valid for any initial configuration of Config-O with infinite number of reds.

4 Conclusion

This paper has considered two simple rules of learning by imitation by which (extremely) boundedly rational agents can learn from their own experiences and the experience of others in their neighbourhood on the integers. The two rules are very similar; the basic difference is that in the first case an agent who succeeds does not want to "fix what isn't broken" and does not change his action, while in the second each agent, no matter what the realisation of his or her own experiment, takes into account the experiences of neighbours in deciding what to do next. We are able to prove that the first rule leads to diffusion of the better technology with probability 1; the second, however, converges in the same way if the better technology is sufficiently better.

We initially began this paper looking at two-dimensional lattices. However, we do not know if a similar result holds in this case. This is a topic for future work. There is some evidence from simulations that clusters of the inferior technology might survive, but the simulations were run for too brief a period of time and on too small a finite grid for us to believe that these results will carry over. Our conjecture is that a result similar to the one we have in this paper holds for two dimensions as well, but this remains to be shown.

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