# Hypergeometric identities associated with statistics on words

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*From the first two authors to the third in honor of his 80th birthday*

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# 1 Introduction

The purpose of this note is to show how combinatorial arguments can produce nontrivial identities between hypergeometric $q$-series in two variables. This will be illustrated by using as examples

1. the major index of a binary word
2. the Durfee square size of an integer partition
3. the number of inversions in a binary word
4. the number of descents in a binary word
5. the sum of the positions of the 0's in a bitstring
6. "lecture hall" statistics on words.

Let $w$ be a word of length $n$ over the alphabet $\{0, 1\}$ (a *binary word*). By the *major index* of $w$ we mean the sum of those indices $j$, $1 \leq j \leq n-1$, for which $w_j > w_{j+1}$, i.e., for which $w_j = 1$ and $w_{j+1} = 0$. Let $f(n,m)$ denote the number of binary words of length $n$ whose major index is $m$ ($f(0,0) = 1$). In Sections 2 and 3, we find the generating function $F(x,q) = \sum_{n,m} f(n,m)x^n q^m$ in various ways, compare it to the known Mahonian form of this function, and thereby obtain an interesting chain of seven equalities, namely

$$\begin{aligned}
F(x,q) &\stackrel{\text{def}}{=} \sum_{n,m\geq 0} f(n,m)x^n q^m &(1)\\
&= \sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q x^n &(2)\\
&= \sum_{n\geq 0} \frac{x^n}{(x;q)_{n+1}} &(3)\\
&= -1 + \sum_{j\geq 0} (1 + (1-2x)q^j) \left( \frac{x^j q^{\binom{j}{2}}}{(x;q)_{j+1}} \right)^2 &(4)\\
&= \sum_{j\geq 0} \left( \frac{x^j q^{j^2/2}}{(x,q)_{j+1}} \right)^2 &(5)\\
&= 1 + \sum_{j\geq 0} \frac{x^{j+1}(1+q^j)}{(x;q)_{j+1}} &(6)\\
&= 1 + 2x + (3+q)x^2 + (4 + 2q + 2q^2)x^3 + \ldots . &(7)
\end{aligned}$$

in which the $[\ ]_q$'s are the Gaussian binomial coefficients.

In Section 2.5 we highlight the connections between $F(x,q)$ and some third order mock theta functions.

Section 4 deals with words over larger alphabets. In Section 5, a related identity is derived by considering the positions of 0's in a bitstring. In Section 6 we look at identities arising from some novel statistics on words. In Section 7, we consider the process of deriving the generating function $F(x,q) = \sum_{n,k\geq 0} t(n,k)x^n q^k$ when a nice product form for the $q$-series $\sum_{k\geq 0} t(n,k)q^k$ is known. We show in this case how $F(x,q)$ can be expressed in terms of statistics on words.

# 2 The equivalence of (1) through (5)

For a binary word $w$ of length $n$, the *blocks* of $w$ are the maximal contiguous subwords whose letters are all the same. The word $w = 11011000$, for example, contains four blocks, namely

11, 0, 11, 000, of lengths 2, 1, 2, 3. The major index of $w$ is then the sum of the indices of the final letters of the blocks of 1's, excepting only a terminal block of 1's. The word $w$ above has major index 2+5=7.

## 2.1 Proof of (1) = (2)

This is simply the assertion that the major index is a Mahonian statistic on words.

## 2.2 Proof of (3)

### 2.2.1 Via generatingfunctionology

The $q$-binomial coefficients satisfy the recurrence

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \qquad (n \geq 0).$$

Let's find their vertical generating function

$$\phi_k(t) \stackrel{\text{def}}{=} \sum_{n \geq 0} t^n \begin{bmatrix} n \\ k \end{bmatrix}_q \qquad (k = 0, 1, 2, \dots).$$

We find that

$$(1 - tq^k)\phi_k(t) = t\phi_{k-1}(t) \qquad (k \geq 1; \phi_0(t) = 1/(1-t)),$$

and therefore

$$\phi_k(t) = \frac{t^k}{\prod_{j=0}^{k}(1 - tq^j)} \qquad (k = 0, 1, 2, \dots).$$

Next, the horizontal generating function (= the Gaussian polynomial)

$$\psi_n(x) \stackrel{\text{def}}{=} \sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

satisfies

$$\psi_{n+1}(x) = \psi_n(qx) + x\psi_n(x) \qquad (n \geq 0; \psi_0 = 1).$$

If we introduce the two variable generating function $\Phi(t,x) = \sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q t^n x^k$, then we find that

$$\Phi(t,x)(1-xt) = t\Phi(t,qx) + 1,$$

which leads to

$$\Phi(t,x) \stackrel{\text{def}}{=} \sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q t^n x^k = \sum_{n\geq 0} \frac{t^n}{\prod_{j=0}^{n}(1-q^j xt)},$$

as required.

#### 2.2.2 Via $q$-series

In [2, Thm. 3.3], (3) is derived from (2) using Cauchy's Theorem [2, Thm. 2.1]:

$$\sum_{k\geq 0} \frac{(a;q)_k x^k}{(q;q)_k} = \prod_{k=0}^{\infty} \frac{(1-axq^k)}{(1-xq^k)},$$

with $a = q^{n+1}$, after setting $n = n+k$ in (2). In the process we have

$$\sum_{k\geq 0} \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k = \prod_{k=0}^{\infty} \frac{(1-xq^{k+n+1})}{(1-xq^k)} = \frac{1}{(x;q)_{n+1}}, \tag{8}$$

the $q$-binomial theorem.

### 2.3 Proof of (1) = (4)

To solve the word problem posed in Section 1, we split it into four cases, namely words with an even (resp. odd) number of blocks, the first of which is a block of 1's (resp. 0's). We will show all steps of the solution for the first case, and then merely exhibit the results for the other three cases.

Let's do the case of words $w$, of length $n$, which have an even number, $2k$, say, of blocks, the first of which is a block of 1's, and suppose that the lengths of these blocks are $a_1, a_2, \ldots, a_{2k}$ (all $a_i \geq 1$). Such a word has descents at the indices $a_1, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + \cdots + a_{2k-1}$, so its major index is

$$\begin{aligned} \mathrm{maj}(w) &= ka_1 + (k-1)a_2 + (k-1)a_3 + \cdots + a_{2k-2} + a_{2k-1} \\ &= \sum_{j=1}^{2k-1} a_{2k-j} \left\lceil \frac{j}{2} \right\rceil. \end{aligned}$$

It follows that the contribution of all the words whose form is that of the first of the four cases is

$$
\begin{aligned}
F_1(x,q,t) &= \sum x^{|w|} q^{\mathrm{maj}(w)} t^{\mathrm{Blocks}(w)} \\
&= \sum_{k\geq 1} \sum_{a_1,\ldots,a_{2k}\geq 1} x^{\sum_{j=1}^{2k} a_j} q^{\sum_{j=1}^{2k-1} a_{2k-j}\lceil j/2\rceil} t^{2k} \\
&= \sum_{k=1}^{\infty} \frac{x^{2k} q^{k^2} t^{2k}}{(1-x)(1-xq^k)\prod_{j=1}^{k-1}(1-xq^j)^2} \\
&= x^2t^2q + x^3\left(t^2q^2+t^2q\right) + x^4\left(t^4q^4+t^2q^3+t^2q^2+t^2q\right) + \ldots .
\end{aligned}
$$

Similarly, in the second case, where the number of blocks is even but the first block consists of 0's, we have

$$
\begin{aligned}
F_2(x,q,t) &= \sum x^{|w|} q^{\mathrm{maj}(w)} t^{\mathrm{Blocks}(w)} \\
&= \sum_{k\geq 1} \sum_{a_1,\ldots,a_{2k}\geq 1} x^{\sum_{j=1}^{2k} a_j} q^{\sum_{j=2}^{2k-1} a_{2k-j}\lceil (j-1)/2\rceil} t^{2k} \\
&= \sum_{k\geq 1} \frac{x^{2k} q^{k(k-1)} t^{2k}}{\prod_{j=0}^{k-1}(1-xq^j)^2} \\
&= t^2x^2 + 2t^2x^3 + x^4(3t^2+t^4q^2) + x^5(4t^2+2t^4q^2+2t^4q^3) + \ldots
\end{aligned}
$$

In the third case the number of blocks is odd, say $2k+1$, with $k\geq 0$, and the first block is all 1's. The major index of such a word is

$$
\mathrm{maj}(w) = \sum_{j=1}^{2k-1} a_{2k-j}\left\lceil \frac{j}{2} \right\rceil .
$$

Thus,

$$
\begin{aligned}
F_3(x,q,t) &= \sum x^{|w|} q^{\mathrm{maj}(w)} t^{\mathrm{Blocks}(w)} \\
&= \sum_{k\geq 0} \sum_{a_1,\ldots,a_{2k+1}\geq 1} x^{\sum_{j=1}^{2k+1} a_j} q^{\sum_{j=1}^{2k-1} a_{2k-j}\lceil j/2\rceil} t^{2k+1} \\
&= \sum_{k\geq 0} \frac{x^{2k+1} q^{k^2} t^{2k+1}}{(1-xq^k)\prod_{j=0}^{k-1}(1-xq^j)^2} \\
&= tx + tx^2 + x^3\left(qt^3+t\right) + x^4\left(q^2t^3+2qt^3+t\right) \\
&\quad + x^5\left(q^4t^5+q^3t^3+2q^2t^3+3qt^3+t\right) + \ldots
\end{aligned}
$$

Finally, if there are $2k+1$ blocks in the word $w$ and the first block is all 0's, the major index is

$$\operatorname{maj}(w) = \sum_{j=0}^{2k-1} a_{2k-j} \left\lceil \frac{j+1}{2} \right\rceil ,$$

so

$$\begin{aligned}
F_4(x,q,t) &= \sum x^{|w|} q^{\operatorname{maj}(w)} t^{\operatorname{Blocks}(w)} \\
&= \sum_{k\geq 0} x^{\sum_{j=1}^{2k+1} a_j} q^{\sum_{j=0}^{2k-1} a_{2k-j} \left\lceil \frac{j+1}{2} \right\rceil} t^{2k+1} \\
&= (1-x) \sum_{k\geq 0} \frac{x^{2k+1} q^{k(k+1)} t^{2k+1}}{\prod_{j=0}^{k} (1-xq^j)^2} \\
&= tx + tx^2 + x^3 \left(t^3y^2 + t\right) + x^4 \left(2t^3y^3 + t^3y^2 + t\right) \\
&\quad + x^5 \left(t^5y^6 + 3t^3y^4 + 2t^3y^3 + t^3y^2 + t\right) + \ldots .
\end{aligned}$$

Now we compute the desired generating function $F(x,q,t)$ as

$$F(x,q,t) = 1 + \sum_{i=1}^{4} F_i(x,q,t)$$

in which the $F_i$ are explicitly shown above. If we put $t=1$ we find that

$$\begin{aligned}
\sum x^{|w|} q^{\operatorname{maj}(w)} &= 1 + 2x + x^2(q+3) + x^3 \left(2q^2 + 2q + 4\right) + x^4 \left(q^4 + 3q^3 + 4q^2 + 3q + 5\right) \\
&\quad + x^5 \left(2q^6 + 2q^5 + 6q^4 + 6q^3 + 6q^2 + 4q + 6\right) + \ldots
\end{aligned}$$

Observe that if we put $q := 1$, the coefficient of each $x^n$ is indeed $2^n$.

On the other hand, the maj statistic is well known to be Mahonian, which implies that its distribution function is

$$\sum_{w} x^{|w|} q^{\operatorname{maj}(w)} = \sum_{n,k} \begin{bmatrix} n \\ k \end{bmatrix}_q x^n,$$

in which the $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are the usual Gaussian polynomials.

It follows that

$$\begin{aligned}
\sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q x^n &= 1 + F_1(x,q,1) + F_2(x,q,1) + F_3(x,q,1) + F_4(x,q,1) \\
&= 1 + \sum_{k=1}^{\infty} \frac{x^{2k} q^{k^2}}{(1-x)(1-xq^k)\prod_{j=1}^{k-1}(1-xq^j)^2} + \sum_{k\geq 1} \frac{x^{2k} q^{k(k-1)}}{\prod_{j=0}^{k-1}(1-xq^j)^2} \\
&\quad + \sum_{k\geq 0} \frac{x^{2k+1} q^{k^2}}{(1-xq^k)\prod_{j=0}^{k-1}(1-xq^j)^2} + (1-x)\sum_{k\geq 0} \frac{x^{2k+1} q^{k(k+1)}}{\prod_{j=0}^{k}(1-xq^j)^2} \\
&= 1 + \sum_{k\geq 1} \frac{x^{2k} q^{k^2}}{(x;q)_k^2} \left( \frac{1-x}{1-xq^k} + \frac{1}{q^k} \right) + \sum_{k\geq 0} \frac{x^{2k+1} q^{k^2}}{(x;q)_k^2} \left( \frac{1}{1-xq^k} + \frac{(1-x)q^k}{(1-xq^k)^2} \right) \\
&= -1 + \sum_{k\geq 0} \frac{(1+(1-2x)q^k)}{(1-xq^k)^2} \left( \frac{x^k q^{\binom{k}{2}}}{(x;q)_k} \right)^2,
\end{aligned}$$

as claimed.

## 2.4 Proof of (5)

We prove (5) in four different ways.

### 2.4.1 Equivalence of (3) and (5) using the Rogers-Fine identity

The Rogers-Fine identity is [5], [4, p. 223]:

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n (\alpha\tau q/\beta;q)_n \beta^n \tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n (\tau;q)_{n+1}}. \tag{9}$$

Setting $\alpha = 0$, $\tau = x$, and $\beta = xq$ in (9) gives

$$\sum_{n=0}^{\infty} \frac{1}{(xq;q)_n} x^n = \sum_{n=0}^{\infty} \frac{x^{2n} q^{n^2}}{(xq;q)_n (x;q)_{n+1}}.$$

Multiply through by $1/(1-x)$ and use the equivalence of (1) and (3) to conclude

$$F(x,q) = \sum_{n=0}^{\infty} \frac{x^n}{(x;q)_{n+1}} = \sum_{n=0}^{\infty} \left( \frac{x^n q^{n^2/2}}{(x;q)_{n+1}} \right)^2.$$

In this form the generating function appears quite similar to, but not identical with (4), though it is of course identical. Consequently, by comparing the two forms, we see that we have proved the small identity

$$\sum_{k\geq 0}\left(\frac{x^k q^{\binom{k}{2}}}{(x,q)_{k+1}}\right)^2 (1-2xq^k) = 1.$$

We show in the following subsection how to transform (4) into (5).

### 2.4.2 Direct proof of (4) = (5)

We would like to prove:

$$-1+\sum_{k\geq 0}(1+(1-2x)q^k)\left(\frac{x^k q^{\binom{k}{2}}}{(x;q)_{k+1}}\right)^2 \quad = \quad \sum_{k\geq 0}\left(\frac{x^k q^{k^2/2}}{(x,q)_{k+1}}\right)^2.$$

Using the fact that

$$1+(1-2x)q^k \quad = \quad -x^2q^{2k}+(1-xq^k)(1-xq^k)+q^k,$$

we can transform as follows:

$$\begin{aligned}
&-1+\sum_{k\geq 0}(1+(1-2x)q^k)\left(\frac{x^k q^{\binom{k}{2}}}{(x;q)_{k+1}}\right)^2\\
&\qquad = -1-\sum_{k\geq 0}\frac{x^{2k+2}q^{k^2+k}}{(x;q)_{k+1}^2}+\sum_{k\geq 0}\frac{x^{2k}q^{k^2-k}}{(x;q)_k^2}+\sum_{k\geq 0}\frac{x^{2k}q^{k^2}}{(x;q)_{k+1}^2}\\
&\qquad = -1-\sum_{k\geq 1}\frac{x^{2k}q^{k^2-k}}{(x;q)_k^2}+\sum_{k\geq 0}\frac{x^{2k}q^{k^2-k}}{(x;q)_k^2}+\sum_{k\geq 0}\frac{x^{2k}q^{k^2}}{(x;q)_{k+1}^2}\\
&\qquad = \sum_{k\geq 0}\frac{x^{2k}q^{k^2}}{(x;q)_{k+1}^2}
\end{aligned}$$

□

### 2.4.3 Equivalence of (1) and (5) by recurrence

As an alternative, we can derive (5) directly from the definition of $F(x, q)$ in terms of binary words.

**Lemma 1** *Let $f(n, m)$ denote the number of binary words of length $n$ whose major index is $m$. Then*

$$f(n, m) = 2f(n-1, m) - f(n-2, m) + f(n-2, m-n+1) \qquad (n \geq 2; m \geq 0) \qquad (10)$$

*with initial conditions $f(0, m) = \delta_{m,0}$, $f(1, m) = 2\delta_{m,0}$.*

**Proof.** Let $S(n, m)$ be the set of binary words of length $n$ with major index $m$, so that $f(n, m) = |S(n, m)|$. Let "$\cdot$" denote concatenation of words and observe that

$$\begin{aligned} \mathrm{maj}(w \cdot 1) &= \mathrm{maj}(w), \\ \mathrm{maj}(w \cdot 10) &= \mathrm{maj}(w) + |w \cdot 1|, \\ \mathrm{maj}(w \cdot 00) &= \mathrm{maj}(w \cdot 0). \end{aligned}$$

Thus

$$\begin{aligned} w \cdot 1 \in S(n, m) &\leftrightarrow w \in S(n-1, m), \\ w \cdot 10 \in S(n, m) &\leftrightarrow w \in S(n-2, m-(n-1)), \\ w \cdot 00 \in S(n, m) &\leftrightarrow w \cdot 0 \in S(n-1, m) - S(n-2, m) \cdot 1. \end{aligned}$$

Since every element of $S(n, m)$ falls into exactly one of the cases above, the result follows. $\square$

As in (1), we define the generating function $F(x, q) = \sum_{n,m \geq 0} f(n, m) x^n q^m$. Next we multiply each of the four terms in (15) by $x^n q^m$ and sum over $n \geq 2$ and $m \geq 0$.

The first term yields $F(x, q) - 2x - 1$, the second gives $2x(F(x, q) - 1)$, the third becomes $x^2 F(x, q)$, and the fourth yields $x^2 q F(xq, q)$. Therefore we have the functional equation

$$F(x, q) = \frac{1 + x^2 q F(xq, q)}{(1-x)^2},$$

whose solution is

$$F(x, q) = \sum_{j \geq 0} \frac{x^{2j} q^{j^2}}{\prod_{\ell=0}^{j} (1 - xq^\ell)^2}.$$

### 2.4.4 Equivalence of (2) and (5) via partitions

We can also give a direct proof of the identity

$$\sum_{n,k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix}_q x^n = \sum_{j\geq 0} \frac{x^{2j} q^{j^2}}{((x;q)_{j+1})^2},$$

using partitions. We'll see the value of this after we look at inversions in Section 3.

We show that both sides count, for every pair $(a, b)$, the number of partitions $\lambda$ in an $a \times b$ box, where $q$ keeps track of $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_a$ and $x$ keeps track of $a + b$. The left-hand side counts all the partitions for fixed $(a, b)$ and then sums over all $(a, b)$. The right-hand side counts all the partitions with Durfee square size $j$, for every $(j + s) \times (j + t)$ box containing them, and then sums over all $j$. $\square$

Let $P(a, b)$ be the set of partitions whose Ferrers diagram fit in an $a \times b$ box. Let $D(\lambda)$ denote the size of the Durfee square of $\lambda$. The argument above actually shows that

$$\sum_{a,b,\geq 0} \sum_{\lambda\in P(a,b)} q^{\lambda} x^{a+b} z^{D(\lambda)} = \sum_{j\geq 0} \frac{x^{2j} q^{j^2}}{((x;q)_{j+1})^2} z^j.$$

We'll return to this at the end of Section 3.

## 2.5 Mock theta functions

It was observed in [3] that there is a connection between $F(x, q)$, defined by (1) - (7), and the following two of Ramanujan's third order mock theta functions ([11], cf. p. 62):

$$f(q) = \sum_{j\geq 0} \frac{q^{j^2}}{(-q,q)_j^2}; \tag{11}$$

$$\omega(q) = \sum_{j\geq 0} \frac{q^{2j^2+2j}}{(q,q^2)_{j+1}^2}. \tag{12}$$

Specifically, appealing to (5), note that

$$F(-1, q) = f(q)/4; \tag{13}$$

$$F(q, q^2) = \omega(q). \tag{14}$$

One of the goals of the paper [3] was to develop a methodology for interpreting $q$-series identities in terms of families of partitions, via an appropriate statistic. After deriving the equivalence of (5) and (3), the appropriate partition statistic was revealed for interpreting $F(x,q)$:

$$\frac{F(x,q)}{1-x} = \sum_{\lambda} q^{|\lambda|} x^{\rho(x)},$$

where the sum is over all partitions, $\lambda$, and the statistic $\rho(\lambda)$ is the sum of the number of parts of $\lambda$ and the largest part of $\lambda$. Note that this is equivalent to the interpretation of $F(x,q)$ in the preceding subsection. This was then combined with the observations (13) and (14) to interpret the mock theta functions (11) and (12) as generating functions for certain families of partitions.

In view of (1), (13), and (14), we see that the mock theta functions (11) and (12) can be interpreted in terms of statistics on binary words as:

$$\begin{aligned} f(q) &= \sum_{w} (-1)^{|w|} q^{\mathrm{m}aj}; \\ \omega(q) &= \sum_{w} q^{|w|+2\mathrm{m}aj}, \end{aligned}$$

where the sum is over all binary words $w$ and $|w|$ denotes the length of $w$.

# 3 An "inversions" view of (5) and (6)

We obtain another identity by carrying out the same sort of analysis on the inversions of a word, rather than the major index. An inversion in a word $w$ is a pair $(i,j)$ such that $i<j$ but $w_i > w_j$ and $\mathrm{inv}(w)$ is the number of inversions in $w$. The statistic inv is also Mahonian on binary words [8], so its distribution is given by (2).

## 3.1 Proof of (6)

Let $f(n,k,m)$ be the number of binary strings of length $n$, containing exactly $k$ 1's, and with $m$ inversions. Then evidently

$$f(n,k,m) = f(n-1,k-1,m) + f(n-1,k,m-k),$$

for $n \geq 2$, with $f(1,k,m) = \delta_{k,0}\delta_{m,0} + \delta_{k,1}\delta_{m,0}$. If we define the generating function $F(x,y,z) = \sum_{n\geq 1, k\geq 0, m\geq 0} f(n,k,m)x^n y^k z^m$, then we find the functional equation

$$F(x,y,z) = \frac{x(1+y) + xF(x,yz,z)}{1-xy},$$

whose solution is

$$F(x,y,z) = \sum_{m\geq 1} \frac{x^m(1+yz^{m-1})}{\prod_{j=0}^{m-1}(1-xyz^j)}.$$

We can now set $y=1$ and find that the number of binary words of length $n$ with $m$ inversions is equal to the coefficient of $x^n q^m$ in

$$\sum_{m\geq 0} \frac{x^{m+1}(1+q^m)}{(x;q)_{m+1}} = 2x + (3+q)x^2 + (4+2q+2q^2)x^3 + \dots.$$

### 3.2 The equivalence of (5) and (6)

Let $g(n,m)$ be the number of binary words of length $n$ with $m$ inversions. The previous subsection showed that (6) is the generating function for $\sum_{n\geq 0, m\geq 0} g(n,m)x^n q^m$.

Because of the equidistribution of maj and inv, $g(n,m) = f(n,m)$, for $f(n,m)$ defined in Section 1. But supposing we didn't know that, we show that $g(n,m)$ satisfies the same recurrence as $f(n,m)$ in Lemma 1 of Section 2.4.3, and therefore it has the same functional equation, whose solution was shown there to be (5).

**Claim**: We have the recurrence

$$g(n,m) = 2g(n-1,m) - g(n-2,m) + g(n-2,m-n+1) \qquad (n\geq 2; m\geq 0) \tag{15}$$

with initial data $g(0,m) = \delta_{m,0}$, $g(1,m) = 2\delta_{m,0}$.

**Proof.** Let $R(n,m)$ be the set of binary words of length $n$ with $m$ inversions, so that $g(n,m) = |R(n,m)|$. Observe that

$$\begin{aligned} \mathrm{inv}(1\cdot w\cdot 0) &= \mathrm{inv}(w) + |w| + 1, \\ \mathrm{inv}(0\cdot w) &= \mathrm{inv}(w), \\ \mathrm{inv}(w\cdot 1) &= \mathrm{inv}(w) \end{aligned}$$

Words of the form $0\cdot w\cdot 1$ fall into both of the last two classes above and all other words fall into exactly one of the three classes above. So,

$$|R(n,m)| = |1\cdot R(n-2,m-(n-1))\cdot 0| + |0\cdot R(n-1,m)| + |R(n-1,m)\cdot 1| - |0\cdot R(n-1,m)\cdot 1|,$$

and the recurrence follows. $\square$

## 3.3 Revisiting (5)

Recall the notation $P(a,b)$, $D(\lambda)$, and $|\lambda|$ from Section 2.4.4 on partitions. View a binary word as a lattice path, where "1" is an east step and "0" is a north step. Then a binary word $w$ with $a$ 0's and $b$ 1's forms the lower boundary of a partition $\lambda \in P(a,b)$. It is not hard to check that

$$\mathrm{inv}(w) = |\lambda|,$$

But also, the Durfee square size, $D(\lambda)$, is interesting, in the following way.

Let $\phi$ be Foata's "second fundamental transformation" on words [6]. When restricted to binary words $w$, $\phi(w)$ is a permutation of $w$, with

$$\mathrm{maj}(w) \quad = \quad \mathrm{inv}(\phi(w)),$$

and $\phi$ proves bijectively that for any $a, b$, maj and inv have the same distribution over the binary words with $a$ 0's and $b$ 1's,

Furthermore, if $\lambda$ is the partition defined by the lattice path associated with $\phi(w)$, then it was shown in [9] that

$$\mathrm{des}(w) \quad = \quad D(\lambda),$$

where $\mathrm{des}(w)$ is the number of descents of $w$. Thus, $(\mathrm{maj}, \mathrm{des})$ and $(\mathrm{inv}, D)$ have the same joint distribution.

We can combine these observations with the identity from the end of Section 2.2.4:

$$\sum_{a,b,\geq 0} \sum_{\lambda \in P(a,b)} q^{\lambda} x^{a+b} z^{D(\lambda)} = \sum_{j \geq 0} \frac{x^{2j} q^{j^2}}{((x;q)_{j+1})^2} z^j$$

to get

$$\begin{aligned} \sum_{j \geq 0} \frac{x^{2j} q^{j^2}}{((x;q)_{j+1})^2} z^j \quad &= \quad \sum_{a,b,\geq 0} \sum_{\lambda \in P(a,b)} q^{\lambda} x^{a+b} z^{D(\lambda)} \\ &= \quad \sum_{w} q^{\mathrm{inv}(w)} x^{|w|} z^{D(\lambda(w))} \\ &= \quad \sum_{w} q^{\mathrm{maj}(w)} x^{|w|} z^{\mathrm{des}(w)}. \end{aligned}$$

So, "des" is something like the "Blocks" statistic used in Section 2.3. However, observe that "des" gives rise to (5), whereas "Blocks" gives rise to (4).

# 4 Larger alphabets

The above results were all obtained by studying binary words. Now let's look at words over the $M$-letter alphabet $[M] = \{0, 1, 2, \ldots, M-1\}$.

Let $f(k_0, k_1, \ldots, k_{M-1}; \mu)$ denote the number of words over $[M]$ that contain exactly $k_0$ 0's, $k_1$ 1's,...,$k_{M-1}$ $M-1$'s, and which have major index $\mu$. Of course the length of such a word is $N = \sum_i k_i$. It is known that major index is Mahonian on this set of words [8] and therefore its distribution is given by the $q$-multinomial coefficient

$$\sum_{\mu \geq 0} f(k_0, k_1, \ldots, k_{M-1}; \mu) q^{\mu} = \begin{bmatrix} N \\ k_0, k_1, \ldots, k_{M-1} \end{bmatrix}_q .$$

See Sloane's sequences A129529, A129531 for the cases $M = 3, 4$. So, if $[M]^*$ denotes the set of all words over $[M]$,

$$F(x, q) = \sum_{w \in [M]^*} q^{\mathrm{maj}(w)} x^{|w|} = \sum_{N \geq 0} \sum_{k_0 + \cdots + k_{M-1} = N} \begin{bmatrix} N \\ k_0, k_1, \ldots, k_{M-1} \end{bmatrix}_q x^N. \tag{16}$$

Rewriting the last expression and applying (8), we find

$$\begin{aligned} F(x,q) &= \sum_{k_0, k_1, \ldots, k_{M-1} \geq 0} \begin{bmatrix} k_0 + \cdots + k_{M-1} \\ k_0, \ldots, k_{M-1} \end{bmatrix}_q x^{k_0 + \cdots + k_{M-1}} \\ &= \sum_{k_0, k_1, \ldots, k_{M-2} \geq 0} \begin{bmatrix} k_0 + \cdots + k_{M-2} \\ k_0, \ldots, k_{M-2} \end{bmatrix}_q x^{k_0 + \cdots + k_{M-2}} \sum_{k_{M-1} \geq 0} \begin{bmatrix} k_0 + \cdots + k_{M-1} \\ k_{M-1} \end{bmatrix}_q x^{k_{M-1}} \\ &= \sum_{k_0, k_1, \ldots, k_{M-2} \geq 0} \begin{bmatrix} k_0 + \cdots + k_{M-2} \\ k_0, \ldots, k_{M-2} \end{bmatrix}_q \frac{x^{k_0 + \cdots + k_{M-2}}}{(x;q)_{k_0 + \cdots + k_{M-2}}}. \end{aligned}$$

This generalizes the equivalence of (2) and (3) which is the $M = 2$ case.

We will consider a variation and get a $q$-difference equation.

Let $f_i(k_0, k_1, \ldots, k_{M-1}; \mu)$ denote the number of words over $[M]$ that contain exactly $k_0$ 0's, $k_1$ 1's,...,$k_{M-1}$ $M-1$'s, and which have major index $\mu$, and whose last letter is $i$ ($i = 0, \ldots, M-1$).

Of these $f_i(k_0, k_1, \ldots, k_{M-1}; \mu)$ words, the number whose penultimate letter is $j$ is

$$\begin{cases} f_j(k_0, k_1, \ldots, k_i - 1, \ldots, k_{M-1}; \mu - (N-1)), & \text{if } j > i, \\ f_j(k_0, k_1, \ldots, k_i - 1, \ldots, k_{M-1}; \mu), & \text{if } j \leq i. \end{cases}$$

Consequently, for $i = 0 \ldots, M-1$, we have

$$\begin{aligned} f_i(k_0, k_1, \ldots, k_{M-1}; \mu) &= \sum_{j>i} f_j(k_0, k_1, \ldots, k_i - 1, \ldots, k_{M-1}; \mu - (N-1)) \\ &\qquad + \sum_{j \le i} f_j(k_0, k_1, \ldots, k_i - 1, \ldots, k_{M-1}; \mu). \end{aligned}$$

Now sum both sides over all $\mathbf{k}$ such that $k_0 + \cdots + k_{M-1} = N$, and write $F_i(N, \mu)$ for $\sum_{k_0+\cdots+k_{M-1}=N} f_i(k_0, k_1, \ldots, k_{M-1}; \mu)$. We obtain

$$F_i(N, \mu) = \sum_{j>i} F_j(N-1, \mu - N + 1) + \sum_{j \le i} F_j(N-1, \mu),$$

with $F_i(1, \mu) = M\delta_{\mu,0}$. In terms of the generating functions

$$\Phi_{N,i} = \sum_{\mu} F_i(N, \mu) q^{\mu},$$

we find that

$$\Phi_{N,i} = q^{N-1} \sum_{j>i} \Phi_{N-1,j} + \sum_{j \le i} \Phi_{N-1,j},$$

with $\Phi_{1,i} = 1$ for all $i = 0, \ldots, M-1$.

Finally, if $\Phi_i(x, q) = \sum_{N \ge 1} \Phi_{N,i} x^N$, we find that

$$\Phi_i(x, q) = x + x \sum_{j>i} \Phi_j(qx, q) + x \sum_{j \le i} \Phi_j(x, q). \qquad (i = 0, 1, \ldots, M-1)$$

# 5 A related identity based on the positions of 0's in bitstrings

If $w$ is a binary string of length $n$, let $\sigma(w)$ be the sum of the positions that contain 0 bits, the positions being labeled $1, 2, \ldots, n$. Thus $f(10101) = 2 + 4 = 6$. We consider the generating function

$$F(x, q) = \sum_{w} x^{|w|} q^{\sigma(w)},$$

the sum extending over all binary words of all lengths.

If we let $T(n,k)$ denote the number of words of length $n$ for which $\sigma(w)=k$, then we have the obvious recurrence $T(n,k)=T(n-1,k)+T(n-1,k-n)$. This leads, in the usual way, to the functional equation

$$F(x,q)=\frac{1+xqF(xq,q)}{1-x}, \tag{17}$$

which in turn leads, by iteration, to the explicit expression

$$F(x,q)=\sum_{j\geq 0}\frac{x^j q^{\binom{j+1}{2}}}{(x;q)_{j+1}}. \tag{18}$$

On the other hand it is easy to see that

$$\sum_k T(n,k)q^k=\prod_{\ell=1}^{n}(1+q^\ell), \tag{19}$$

since each position $\ell$ in $w$ can either be 1, which contributes $\ell$ to $\sigma(w)$, or 0, which contributes nothing. Thus, we have the identity

$$\sum_{j\geq 0}\frac{x^j q^{\binom{j+1}{2}}}{(x;q)_{j+1}}=\sum_{n\geq 0}x^n\prod_{\ell=1}^{n}(1+q^\ell). \tag{20}$$

Note that (20) is a specialization of Heine's second transformation (eq. III.2 in Appendix III of [7] with $a=-q$, $b=q$, $c=0$, $z=x$).

## 5.1 A partition theory view

We can interpret the identity (20) in terms of partitions.

We claim that both sides of the identity count all pairs $(\lambda,n)$ where $\lambda$ is a partition into distinct parts and $n$ is greater than or equal to the largest part of $\lambda$.

The right-hand side counts this by summing over all $n$ the terms $x^n q^{|\lambda|}$ for all $\lambda$ into distinct parts the largest of which is $\leq n$.

The left-hand side counts this by summing over all $j$ the terms $x^n q^{|\lambda|}$ for all pairs $(\lambda,n)$ where $\lambda$ is a partition into $j$ positive distinct parts, the largest of which is $\leq n$. To see this, If $\lambda$ is a partition into $j$ distinct positive parts, then subtracting the staircase partition $(j,j-1,\ldots,1)$ from $\lambda$ subtracts $\binom{j+1}{2}$ from the $q$-weight of $\lambda$ and subtracts $j$ from the largest part of $\lambda$, leaving an ordinary partition $\lambda'$ with at most $j$ parts. Such $\lambda'$ are counted in the left-hand-side of (20) by $1/(x;q)_{j+1}$, where $x$ keeps track of the size of the largest part of $\lambda'$ *plus* an excess corresponding to the number of times the "0" part is selected as the $1/(1-x)$ factor in the product.

## 5.2 A generalization

Let $w$ be a word over the $K$ letter alphabet $\{0, 1, \ldots, K-1\}$ and let

$$\sigma(w) = \sum_{i=1}^{n} i w_i.$$

We have $f(10101) = 1+3+5 = 9$ and $f(120301) = 1+4+12+6 = 23$. We consider the generating function

$$F(x,q) = \sum_{w} x^{|w|} q^{\sigma(w)},$$

the sum extending over all $K$-ary words of all lengths.

If we let $T(n,k)$ denote the number of words of length $n$ for which $\sigma(w) = k$, then we have the obvious recurrence

$$T(n,k) = \sum_{i=0}^{K-1} T(n-1, k-in). \qquad (n \geq 1; T(0,k) = \delta_{k,0}).$$

If we take our generating function in the form $F(x,q) = \sum_{k,n\geq 0} T(n,k) x^n q^k$, this leads, in the usual way, to the functional equation

$$F(x,q) = \frac{1}{1-x} + \frac{x}{1-x} \sum_{i=1}^{K-1} q^i F(xq^i, q), \tag{21}$$

In the binary case ($K = 2$), this agrees with (17), which has the explicit expression (18).

On the other hand, since a $j$ in position $\ell$ contributes $j\ell$ to $\sigma(w)$, so

$$\sum_{k} T(n,k) q^k = \prod_{\ell=1}^{n} (1 + q^\ell + q^{2\ell} + \cdots + q^{(K-1)\ell}) = \prod_{\ell=1}^{n} \frac{1-q^{K\ell}}{1-q^\ell}, \tag{22}$$

and in the case $K = 2$ we have another view of the identity (20).

We would like an explicit solution to the functional equation (21) for $K > 2$, analogous to (20). Recall that (20) was a special case of Heine's second transformation. There is no analog of Heine's second transformation for $K > 2$. However, there is an analog of the first Heine transformation that can be applied. We make use of the following, which is Lemma 1 from [1]:

$$\sum_{n\geq 0} \frac{t^n (a;q^k)_n (b;q)_{kn}}{(q^k;q^k)_n (c;q)_{kn}} \;=\; \frac{(b;q)_\infty (at;q^k)_\infty}{(c;q)_\infty (t;q^k)_\infty} \sum_{n\geq 0} \frac{b^n (c/b;q)_n (t;q^k)_n}{(q;q)_n (at;q^k)_n}. \tag{23}$$

Setting $a = c = 0$, $b = x$, $k = K$, and $t = q^k$ in (23) gives

$$F(x,q) \;=\; \sum_{n\geq 0} \frac{x^n (q^K;q^K)_n}{(q;q)_n} \;=\; \frac{(q^K;q^K)_\infty}{(x;q)_\infty} \sum_{n\geq 0} \frac{q^{Kn}(x;q)_{Kn}}{(q^K;q^K)_n}.$$

# 6 "Lecture hall" statistics on words

The following statistics arose in [10] in a more general context, but we specialize them here to words. For a $K$-ary word $w$ of length $n$, define the following statistics:

$$\begin{aligned}
\mathrm{ASC}(\mathrm{w}) &= \{ i \mid i = 0 \text{ and } w_1 > 0 \;\text{ or }\; 1 \leq i < n \text{ and } w_i < w_{i+1} \}; \\
\mathrm{asc}(w) &= |\mathrm{ASC}(w)|; \\
\mathrm{lhp}(w) &= -\mathrm{inv}(w) + \sum_{i \in \mathrm{ASC}(w)} K(n-i);
\end{aligned}$$

It follows from Theorem 5 in [10] that

$$\sum_{t\geq 0} \sum_{\lambda \in P(n,Kt)} q^{|\lambda|} x^t = \frac{\sum_{w \in [K]^n} q^{\mathrm{lhp}(w)} x^{\mathrm{asc}(w)}}{\prod_{i=0}^n (1 - xq^{Ki})},$$

where $[K] = \{0, 1, \ldots, K-1\}$.

As observed in [10], the inner sum on the left is a $q$-binomial coefficient, so we get the identity:

$$\sum_{t\geq 0} \begin{bmatrix} n+Kt \\ n \end{bmatrix}_q x^t = \frac{\sum_{w \in [K]^n} q^{\mathrm{lhp}(w)} x^{\mathrm{asc}(w)}}{\prod_{i=0}^n (1 - xq^{Ki})}.$$

Multiplying both sides by $(1-x)$ and then setting $x = 1$ gives

$$\sum_{w \in [K]^n} q^{\mathrm{lhp}(w)} = \prod_{\ell=1}^{n} (1 + q^\ell + q^{2\ell} + \cdots + q^{(K-1)\ell}),$$

the same distribution as $\sum_i iw_i$ from Section 5.2 (!) We don't have any nice combinatorial explanation for this yet.

Experiments indicate that when $K = 2$, we can actually get the following refinement:

$$\sum_{t\geq 0}\sum_{i=0}^{n} \begin{bmatrix} n+t-i \\ t \end{bmatrix}_{q^2} \begin{bmatrix} t-1+i \\ t-1 \end{bmatrix}_{q^2} (qz)^i x^t = \frac{\sum_{w\in[2]^n} q^{\mathrm{lhp}(w)} x^{\mathrm{asc}(w)} z^{w_1+w_2+\cdots+w_n}}{\prod_{i=0}^{n}(1-xq^{2i})}.$$

We can show that to prove this, it would suffice to verify that the innermost summand on the left is the generating function for partitions in an $n$ by $2t$ box with $i$ odd parts.

# 7 The generating function of the terms of a closed form $q$-series

In trying to find the solution to a combinatorial problem, one often goes through the procedure of finding a recurrence, then a functional equation for the generating function, then by iteration, the solution of that functional equation, and then, with some luck, a nice product form for the coefficients that are of interest.

Here, let's invert that process. Suppose we have a sequence $t(n, k)$ which satisfies

$$\sum_{k\geq 0} t(n,k)q^k = \prod_{j=1}^{n} \frac{a(q^j)}{b(q^j)},$$

where $a(t), b(t)$ are fixed polynomials in $t$. In other words, we suppose that the sum on the left is a $q$-hypergeometric term in $n$. What we would like to know is the generating function

$$F(x, q) = \sum_{n,k} t(n, k)x^n q^k.$$

To do this, put $f(n) = \sum_{k\geq 0} t(n, k)q^k$, and then we have

$$b(q^n)f(n) = a(q^n)f(n-1). \qquad (n \geq 1; f(0) = 1) \tag{24}$$

To simplify the appearance of the following results, let $R$ be the operator that transforms $x$ to $xq$, i.e., $Rf(x) = f(xq)$, and suppose our polynomials $a, b$ are $a(t) = \sum a_j t^j$ and $b(t) = \sum_j b_j t^j$. Further, take the generating function in the form

$$F(x, q) = \sum_{n,k\geq 0} t(n, k)x^n q^k.$$

Now multiply (24) by $x^n$ and sum over $n \geq 1$, to find that

$$(b(R) - xa(qR))F(x, q) = 1 \tag{25}$$

is the functional equation of the generating function.

## 7.1 Examples

**Example 1** In the case (19) above we have $a(t) = 1 + t$ and $b(t) = 1$. The functional equation (25) now reads as

$$(1 - x(1 + qR))F(x, q) = 1 = (1 - x)F(x, q) - xqF(xq, q),$$

in agreement with (17).

**Example 2** Consider the case of the statistic $\sigma(w)$ of Section 5.2 on $K$-ary words when $K = 3$. (This has the same distribution as the statistic lhp from Section 6.) Here we have from (22) that $a(t) = 1 + t + t^2$ and $b(t) = 1$. The functional equation (25) takes the form $F(x, q) = 1 + x(F(x, q) + qF(xq, q) + q^2F(xq^2, q))$, i.e.,

$$F(x, q) = \frac{1}{1 - x}\left(1 + xqF(xq, q) + xq^2F(xq^2, q)\right), \tag{26}$$

in agreement with (21). We see by iteration that the solution of this equation is going to be a sum of terms of the form

$$\frac{q^\alpha x^\beta}{\prod_{i=1}^{n+1}(1 - xq^{s_i})}, \tag{27}$$

for some collection of $\alpha$, $\beta$, $s_i$ to be defined. We want to identify exactly which terms occur. The set $T$ of such terms is defined inductively by the two rules

$$\text{(i)} \quad \frac{1}{1 - x} \in T;$$

and

$$\text{(ii)} \quad \text{if} \quad \frac{q^\alpha x^\beta}{\prod_{i=1}^{n+1}(1 - xq^{s_i})} \in T,$$

then both of the following terms must be in $T$ :

$$\frac{q^{\alpha+\beta+1}x^{\beta+1}}{(1 - x)\prod_{i=1}^{n+1}(1 - xq^{s_i+1})} \quad \text{and} \quad \frac{q^{\alpha+2\beta+2}x^{\beta+1}}{(1 - x)\prod_{i=1}^{n+1}(1 - xq^{s_i+2})}.$$

It is now straightforward to verify that the inductive rules define $T$ to be:

$$T = \left\{ \frac{q^{\sigma(w)} x^{|w|}}{\prod_{i=1}^{|w|+1} (1 - x q^{w_i + \cdots + w_{|w|}})} \;\middle|\; w \in \{1,2\}^* \right\}.$$

The generating function is now

$$F(x,q) = \sum_{w \in \{1,2\}^*} \frac{q^{\sigma(w)} x^{|w|}}{\prod_{i=1}^{|w|+1} (1 - x q^{w_i + \cdots + w_{|w|}})}.$$

Consequently we have the identity

$$\sum_{w \in \{1,2\}^*} \frac{q^{\sigma(w)} x^{|w|}}{\prod_{i=1}^{|w|+1} (1 - x q^{w_i + \cdots + w_{|w|}})} = \sum_{n \geq 0} x^n \prod_{j=1}^{n} (1 + q^j + q^{2j}). \tag{28}$$

We're going to tweak the left side of (28) in the hope of making it prettier.

First we change the alphabet from $\{1,2\}$ to $\{0,1\}$, just because it's friendlier. To do that, define new variables $\{v_i\}_{i=1}^n$ by $v_i = w_i - 1$ $(i = 1, \ldots, n)$, where $n = |w|$. Then the gf becomes

$$\sum_{v \in \{0,1\}^*} \frac{q^{\sigma(w)} x^{|v|}}{\prod_{i=1}^{|v|+1} (1 - x q^{w_i + \cdots + w_n})},$$

where we have temporarily used some $v$'s and some $w$'s.

Now introduce yet another set of variables, namely

$$u_i = w_i + \cdots + w_n = v_i + \cdots + v_n + n - i + 1 \quad (i = 1, \ldots, n).$$

Then we have

$$\sigma(w) = \sum_{i=1}^{n} i w_i = (w_1 + \cdots + w_n) + (w_2 + \cdots + w_n) + \cdots + w_n = u_1 + \cdots + u_n = \Sigma(u),$$

say. The generating function now reads as

$$\sum_{u} \frac{q^{\Sigma(u)} x^{|u|}}{\prod_{i=1}^{|u|+1} (1 - x q^{u_i})}$$

which is now entirely in terms of the $u_i$'s, but we need to clarify the set of vectors $u$ over which the outer summation extends.

Say that a sequence $\{t_i\}_{i=1}^{n+1}$ of nonnegative integers is *slowly decreasing* if $t_{n+1} = 0$, and we have $t_i - t_{i+1} = 1$ or $2$ for all $i = 1, \ldots, n$. Then the outer sum above runs over all slowly decreasing sequences of all lengths, i.e., it is

$$\sum_{u \in \mathrm{sd}} \frac{q^{\Sigma(u)} x^{|u|-1}}{\prod_{i=1}^{|u|}(1 - xq^{u_i})}.$$

where sd is the set of all slowly decreasing sequences, $\Sigma(u)$ is the sum of the entries of $u$, and $|u|$ is the length of $u$ (including the mandatory 0 at the end).

### 7.2 A generalization

In the same way we derived (28), we can use the functional equation (25) to derive the following general result.

Suppose $t(n,k)$ satisfies

$$\sum_{k \geq 0} t(n,k) q^k = \prod_{j=1}^{n} \frac{a(q^j)}{b(q^j)},$$

where $a(t), b(t)$ are fixed polynomials in $t$, $a(t) = \sum_{t=0}^{K-1} a_i t^i$, and $b(t) = \sum_{t=0}^{K-1} b_i t^i$. Then

$$F(x,q) \;=\; \sum_{n,k} t(n,k) x^n q^k = \sum_{w \in \{1,2,\ldots,K-1\}^*} \frac{\prod_{i=1}^{|w|}(a_{w_i} x q^{i w_i} - b_{w_i})}{\prod_{i=1}^{|w|+1}(b_0 - a_0 x q^{w_i + \cdots + w_{|w|}})}.$$

This shows how the statistics $iw_i$ on words arise naturally in $q$-series, with the special case of $\sigma(w)$ appearing when the polynomial $b$ is constant.

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