ON DIFFERENTIATION AND HARMONIC NUMBERS

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Abstract. In a paper of Andrews and Uchimura [AU], it is shown how differentiation applied to hypergeometric identities produces formulas for harmonic and q-harmonic numbers. Here we recall two binomial coefficient sums that appear in [M], and reprove them using the techniques of [AU].

1. Introduction and Goal

It is known that many binomial coefficient formulas follow from hypergeometric series identities [A1]. However, as is demonstrated in [AU] there are binomial coefficient identities which are not necessarily of this type. Here we recall two binomial coefficient sums from [M] and reprove them using the methods of [AU].

We first recall identities from (6.21) and (5.28) in [M]:

$$\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}\binom{n}{j}\big(H_{n+j}+H_{m+j}-2H_j\big)=0, \tag{1.1}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}\binom{n}{j}(1+j(H_{n+j}+H_{m+j}-2H_j)) = (-1)^n(1+n+m), \tag{1.2}$$

where $H_n := 1+\frac{1}{2}+\cdots+\frac{1}{n}$. For both identities we have the condition $n\geq m\geq 1$. In [M], these identities were crucial in proving several Beukers like supercongruences that had been observed numerically by Fernando Rodriguez-Villegas [FRV].

In [M], these identities were broken up into smaller pieces, and each part was evaluated using Wilf-Zeilberger [PWZ] theory. Although these were evaluated exactly, for the goal of [M] one only needed to show that the two sums were congruent to zero modulo p, where $n+m=p-1$ and $p\geq 5$ was prime. Our objective here is two-fold: to show that this can be

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accomplished from differentiation of classical hypergeometric identities and to bring attention to this useful technique found in [AU].

We recall some basic definitions,

$$\delta[f(x)] := f'(0) \text{ and } \binom{x+n+j}{n} := \frac{(x+j+1)_n}{(1)_n}, \tag{1.3}$$

where

$$(a)_n := a(a+1)\cdots(a+n-1). \tag{1.4}$$

Thus we can write

$$\delta\Big[\binom{x+n+j}{n}\Big] = \binom{n+j}{n}(H_{n+j} - H_j), \text{ and } \binom{n}{j} = (-1)^j \frac{(-n)_j}{j!}. \tag{1.5}$$

We also recall the standard notation for hypergeometric series [A2],

$$ {}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \;\Big|\; x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} x^n, \tag{1.6}$$

as well as the Chu-Vandermonde sum [A2]:

$$ {}_2F_1\left(\begin{matrix} -n, & a \\ & c \end{matrix} \;\Big|\; 1\right) = \frac{(c-a)_n}{(c)_n}. \tag{1.7}$$

2. The First Identity

Here we break the second identity up into two parts. For the first part, we proceed as follows:

$$\begin{aligned}
\sum_{j=0}^{n} (-1)^j \binom{n+j}{j}\binom{n}{j}(H_{n+j} - H_j) &= \sum_{j=0}^{n} \frac{(-n)_j}{j!}\binom{n+j}{j}(H_{n+j} - H_j) \\
&= \delta\Big[\sum_{j=0}^{n} \frac{(-n)_j}{j!}\frac{(x+j+1)_n}{n!}\Big] = \delta\Big[\frac{(x+1)_n}{n!}\sum_{j=0}^{n} \frac{(-n)_j}{j!}\frac{(x+n+1)_j}{(x+1)_j}\Big] \\
&= \delta\Big[\frac{(x+1)_n}{n!} \cdot {}_2F_1\left(\begin{matrix} -n, & x+n+1 \\ & x+1 \end{matrix} \;\Big|\; 1\right)\Big] \\
&= \delta\Big[\frac{(x+1)_n}{n!} \cdot \frac{(-n)_n}{(x+1)_n}\Big] = \delta[(-1)^n] = 0
\end{aligned}$$

The second part is analogous, but we replace n with $p-m-1$, and then consider it mod p.

$$\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}\binom{n}{j}(H_{m+j}-H_j) = \sum_{j=0}^{n}\frac{(-n)_j}{j!}\frac{(n+1)_j}{j!}(H_{m+j}-H_j)$$
$$= \sum_{j=0}^{n}\frac{(m-p+1)_j}{j!}\frac{(p-m)_j}{j!}(H_{m+j}-H_j)$$
$$\equiv \sum_{j=0}^{m}\frac{(-m)_j}{j!}\frac{(m+1)_j}{j!}(H_{m+j}-H_j) \pmod p = 0$$

Putting both parts together, it follows that the first identity is congruent to zero mod p.

3. The Second Identity

We break the first identity up into three sums, and analyze each separately. The first is well known, but we include it for the sake of completeness.

$$\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}\binom{n}{j} = \sum_{j=0}^{n}\frac{(n+1)!}{j!}\frac{(-n)_j}{j!}$$
$$= {}_2F_1\left(\begin{matrix}-n, & n+1\\ & 1\end{matrix}\ \Big|\ 1\right) = \frac{(-n)_n}{(1)_n} = (-1)^n.$$

For the second sum, we proceed as follows:

$$\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}\binom{n}{j}(j(H_{n+j}-H_j))$$
$$= \delta\Big[\sum_{j=0}^{n}\frac{(-n)_j}{j!}\frac{(x+j+1)_n}{n!}\cdot j\Big] = \delta\Big[\frac{(x+1)_n}{n!}\sum_{j=1}^{n}\frac{(-n)_j}{(j-1)!}\frac{(x+n+1)_j}{(x+1)_j}\Big]$$
$$= \delta\Big[\frac{(x+1)_n}{n!}\frac{(x+n+1)(-n)}{(x+1)}\sum_{j=1}^{n}\frac{(-n+1)_{(j-1)}}{(j-1)!}\frac{(x+n+2)_{(j-1)}}{(x+2)_{(j-1)}}\Big]$$
$$= \delta\Big[\frac{(x+1)_n}{n!}\frac{(x+n+1)(-n)}{(x+1)}\,{}_2F_1\left(\begin{matrix}-n+1, & x+n+2\\ & x+2\end{matrix}\ \Big|\ 1\right)\Big]$$
$$= \delta\Big[\frac{(x+1)_n}{n!}\frac{(x+n+1)(-n)}{(x+1)}\frac{(-n)_{n-1}}{(x+2)_{n-1}}\Big]$$
$$= \delta\Big[\frac{(x+1)_n}{n!}\frac{(x+n+1)(-n)(-1)^{n-1}n!}{(x+1)_n}\Big]$$
$$= \delta\Big[(x+n+1)(-n)(-1)^{n-1}\Big] = (-1)^n\cdot n.$$

for the third and final sum, we recall that $n+m=p-1$ and argue as before

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^j\binom{n+j}{j}\binom{n}{j}(j(H_{m+j}-H_j))\\&\qquad\equiv \delta\Big[\frac{(x+1)_m}{m!}\sum_{j=1}^{m}\frac{(-m)_j}{(j-1)!}\frac{(x+m+1)_j}{(x+1)_j}\Big] \pmod p\\&\qquad=(-1)^m\cdot m=(-1)^n\cdot m\end{aligned}$$

The last equality follows because $n+m=p-1$ is even, thus n and m have the same parity. Therefore the sum is congruent mod p to $(-1)^n(1+n+m)=(-1)^np$, i.e. zero mod p.

4. Remark

The referee speculated as to the truth of the following general identity

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\binom{x+j}{n}\prod_{k=1}^{m}(j-k+1)=\binom{n}{m}(x+1)_m, \tag{4.1}$$

which would shorten the proofs in the previous two sections. Indeed, replace x with $x+n$ and then let let $m=0,\ 1$ in sections 2 and 3 respectively. Proving this identity is an opportunity to emphasize this note's underlying theme: write it as a hypergeometric series and then use Chu-Vandermonde to transform it. Replacing x with $x+n$ for convenience, we have

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{x+n+j}{n}\prod_{k=1}^{m}(j-k+1)\\&=\sum_{j=m}^{n}\frac{(-n)_j}{(j-m)!}\frac{(x+j+1)_n}{n!}=\frac{(x+1)_n}{n!}\sum_{j=m}^{n}\frac{(-n)_j}{(j-m)!}\frac{(x+n+1)_j}{(x+1)_j}\\&=\frac{(x+1)_n}{n!}\frac{(x+n+1)_m(-n)_m}{(x+1)_m}\sum_{j=m}^{n}\frac{(-n+m)_{(j-m)}}{(j-m)!}\frac{(x+n+m+1)_{(j-m)}}{(x+m+1)_{(j-m)}}\\&=\frac{(x+1)_n}{n!}\frac{(x+n+1)_m(-n)_m}{(x+1)_m}{}_2F_1\left(\begin{matrix}-n+m, & x+n+m+1\\ & x+m+1\end{matrix}\ \Big|\ 1\right)\\&=\frac{(x+1)_n}{n!}\frac{(x+n+1)_m(-n)_m}{(x+1)_m}\frac{(-n)_{n-m}}{(x+m+1)_{n-m}}\\&=\frac{(x+n+1)_m}{n!}\frac{(-1)^mn!}{(n-m)!}\frac{(-1)^{n-m}n!}{m!}\\&=(-1)^n\binom{n}{m}(x+n+1)_m.\end{aligned}$$

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