ON THE BROKEN 1-DIAMOND PARTITION

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ABSTRACT. We introduce a crank-like statistic for a different class of partitions. In [AP], Andrews and Paule initiated the study of broken k-diamond partitions. Their study of the respective generating functions led to an infinite family of modular forms, about which they were able to produce interesting arithmetic theorems and conjectures for the related partition functions. Here we establish a crank-like statistic for the broken 1-diamond partition and discuss its role in congruence properties.

1. Introduction and Statement of Results

Among the most celebrated results in the theory of partitions are Ramanujan's congruences for the partition function:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

We note that there are many different proofs and generalizations of these formulas. Dyson [D] provided combinatorial insight into these congruences with a simple statistic called the rank. Although this explains the first two congruences (see Atkin and Swinnerton-Dyer [AS]), it does not explain the third. For the third, Dyson conjectured the existence of an additional statistic, which he called the crank. Forty years after his conjecture was made, Andrews and Garvan ([G], [AG]) defined a function and showed that it does indeed dissect the Ramanujan congruences modulo 11; it also explains the modulo 5 and 7 congruences. As this paper shows, crank-like statistics exist in other partition settings as well.

In [AP] Andrews and Paule, using MacMahon's Partition Analysis, initiated the study of broken k-diamond partitions. Their study of the respective generating functions led to an infinite family of modular forms, about which they produced interesting arithmetic theorems and conjectures for the related partition functions. In this paper, we introduce a new crank-like statistic for this class of partitions. More specifically we consider the broken 1-diamond partition, and motivated by works of Ahlgren and Ono ([O1], [AO]) and Mahlburg [M], we discuss partition congruences associated to this statistic.

To introduce broken 1-diamond partitions, we follow the exposition in [AP], and begin with a basic example of classical plane partitions, treated by MacMahon in [Ma]. Here, the non-negative integer parts a_i of the partitions are placed at the corners of a square such that the

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following order relations are satisfied:

$$a_1 \geq a_2,\ a_1 \geq a_3,\ a_2 \geq a_4,\ \text{and}\ a_3 \geq a_4. \tag{1.1}$$

We use arrows as an alternative description for $\geq$ relations; for instance, Fig. 1 represents the relations (1.1). We interpret an arrow pointing from a_i to a_j as $a_i \geq a_j$.

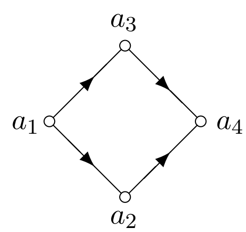

FIGURE 1. The inequalities (1.1)

In [Ma], MacMahon derives the general generating function using partition analysis

$$\begin{aligned} \varphi &:= \sum x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} \\ &= \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}, \end{aligned} \tag{1.2}$$

the sum being over all non-negative integers a_i satisfying (1.1). He further observes that by setting $x_1 = x_2 = x_3 = x_4 = q$, the resulting generating function is

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)}.$$

As in [AP], we now consider the plane partition diamond of length n.

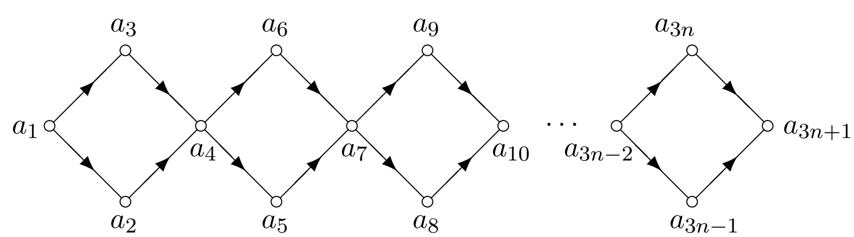

FIGURE 2. A plane partition diamond of length n

Definition 1.1. For $n \geq 1$ define

$$H_n := \{(a_1, \ldots, a_{3n+1}) \in \mathbb{N}^{3n+1} : \text{the } a_i \text{ satisfy the relations in Fig. 2}\},$$

$$h_n := h_n(x_1, \ldots, x_{3n+1}) := \sum_{(a_1,\ldots,a_{3n+1}) \in H_n} x_1^{a_1} x_2^{a_2} \cdots x_{3n+1}^{a_{3n+1}},$$

and $h_n(q) := h_n(q, \ldots, q)$.

In [AP], they prove a generating function in closed form which specializes to

Theorem 1.2. *For $n \geq 1$,*

$$h_n(q) = \frac{\prod_{j=0}^{n-1}(1+q^{3j+2})}{\prod_{j=1}^{3n+1}(1-q^j)}.$$

Plane diamond partitions with a source deleted are then considered.

Definition 1.3. For $n \geq 1$ define

$$H_n^* := \{(a_2, \ldots, a_{3n+1}) \in \mathbb{N}^{3n}\text{: the } a_i \text{ satisfy the relations in Fig. 2 where the vertex labelled } a_1 \text{ has been deleted}\},$$

$$h_n^* := h_n^*(y_2, \ldots, y_{3n+1}) := \sum_{(a_2,\ldots,a_{3n+1}) \in H_n^*} y_2^{a_2} y_3^{a_3} \cdots y_{3n+1}^{a_{3n+1}},$$

and $h_n^*(q) := h_n^*(q, q, \ldots, q)$.

Again, they prove a generating function which specializes to

Theorem 1.4. *For $n \geq 1$,*

$$h_n^*(q) = \frac{\prod_{j=0}^{n-1}(1+q^{3j+1})}{\prod_{j=1}^{3n}(1-q^j)}.$$

Combining these two notions, a broken 1-diamond partition consists of two separate plane partition diamonds of length n, where in one of them the source is deleted.

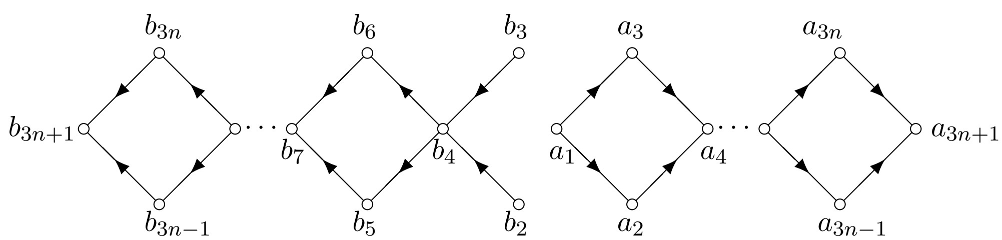

FIGURE 3. A broken 1-diamond of length $2n$

Definition 1.5. For $n \geq 1$ define

$$\begin{aligned} H_n^{\diamondsuit} &:= \{(b_2, \ldots, b_{3n+1}, a_1, a_2, \ldots, a_{3n+1}) \in \mathbb{N}^{6n+1} : \text{the } a_i \text{ and } b_i \\ &\quad\ \text{satisfy all the relations in Fig. 3}\}, \\ h_n^{\diamondsuit} &:= h_n^{\diamondsuit}(y_2, \ldots, y_{3n+1}; x_1, x_2, \ldots, x_{3n+1}) \\ &:= \sum_{(b_2,\ldots,b_{3n+1},a_1,a_2,\ldots,a_{3n+1}) \in H_n^{\diamondsuit}} y_2^{b_2} \cdots y_{3n+1}^{b_{3n+1}} \times x_1^{a_1} x_2^{a_2} \cdots x_{3n+1}^{a_{3n+1}}, \end{aligned}$$

and $h_n^{\diamondsuit}(q) := h_n^{\diamondsuit}(q, q, \ldots, q)$.

One sees that $h_n^{\diamondsuit} = h_n h_n^*$, and given Theorems 1.2 and 1.4, with $n \to \infty$, we have

Theorem 1.6.
$$h_\infty^\Diamond = \prod_{j=1}^\infty \frac{(1+q^j)}{(1-q^j)^2(1+q^{3j})} = \frac{q^{\frac{1}{6}}\ \eta(2\tau)\ \eta\big(3\tau\big)}{\eta(\tau)^3\ \eta\big(6\tau\big)},$$
where $q = e^{2\pi i\tau}$, and $\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^\infty (1-q^n)$ is Dedekind's η-function.

Definition 1.7. For $n \geq 0$, let $\Delta_1(n)$ denote the total number of broken 1-diamond partitions,
$$\sum_{n=0}^\infty \Delta_1(n)q^n := h_\infty^\Diamond.$$

To state our results, more definitions are required. We define the following:

$$\mathcal{D} := \{p \text{ prime} : p \equiv 1, 25, 37, 47, 59, \text{ or } 83 \pmod{84}\}, \tag{1.3}$$

$$\delta_\ell := \tfrac{\ell^2-1}{6},\ \varepsilon_\ell := (\tfrac{-6}{\ell}), \tag{1.4}$$

$$S_\ell := \{\beta \in \{0, 1, \ldots, \ell-1\} : (\tfrac{\beta+\delta_\ell}{\ell}) = 0 \text{ or } -\varepsilon_\ell)\}. \tag{1.5}$$

Motivated by Ahlgren and Ono [AO], we show that for each prime $\ell \in \mathcal{D}$ there is a Ramanujan-type congruence.

Theorem 1.8. *Suppose that $\ell \in \mathcal{D}$, k a positive integer, and $\beta \in S_\ell$. Then a positive proportion of primes $\mathcal{Q} \equiv -1 \pmod{6\ell}$ have the property that,*
$$\Delta_1\left(\frac{\mathcal{Q}n+1}{6}\right) \equiv 0 \pmod{\ell^k}$$
for all $n \equiv 1 - 6\beta \pmod{6\ell}$ that are not divisible by $\mathcal{Q}$.

Corollary 1.9. *Suppose that $\ell \in \mathcal{D}$, k a positive integer, and $\beta \in S_\ell$. Then there are infinitely many non-nested arithmetic progressions $\{An+B\} \subseteq \{\ell n + \beta\}$ such that for every integer n we have*
$$\Delta_1(An+B) \equiv 0 \pmod{\ell^k}.$$

We discuss a statistic and its role in the congruence properties of $\Delta_1(n)$. In [AP], a straightforward proof of the following theorem is given.

Theorem 1.10. *If $n \geq 0$, then $\Delta_1(2n+1) \equiv 0 \pmod 3$.*

Given $\lambda = (b_2, b_3, b_4, \ldots a_1, a_2, a_3, \ldots) \in H_\infty^\Diamond$, we now define a new statistic $\mathcal{R}$ to explain this congruence.

Definition 1.11.
$$\mathcal{R}(\lambda) := \Big[a_1 - \sum_{i=0}^\infty \max(a_{3i+3} - a_{3i+2}, 0)\Big] - \Big[b_2 - \sum_{j=1}^\infty \max(b_{3j+3} - b_{3j+2}, 0)\Big]. \tag{1.6}$$

The number of partitions of n with statistic m is denoted by $\mathcal{R}(m,n)$, and the number of partitions of n with statistic congruent to m modulo N by $\mathcal{R}(m,N,n)$. This statistic then provides a combinatorial proof of Theorem 1.10.

Theorem 1.12. *For $n \geq 0$, $\mathcal{R}(0,3,2n+1) = \mathcal{R}(1,3,2n+1) = \mathcal{R}(2,3,2n+1)$.*

Motivated by Mahlburg [M], we can show that for each prime $\ell \in \mathcal{D}$, there is Ramanujan-type congruence explained by this statistic.

Theorem 1.13. *Suppose that $\ell \in \mathcal{D}$, k and j positive integers, and $\beta \in S_\ell$. Then a positive proportion of primes $\mathcal{Q} \equiv -1 \pmod{6\ell}$ have the property that for every $0 \leq m \leq \ell^j - 1$,*

$$\mathcal{R}\left(m, \ell^j, \frac{\mathcal{Q}n+1}{6}\right) \equiv 0 \pmod{\ell^k}$$

for all $n \equiv 1 - 6\beta \pmod{6\ell}$ that are not divisible by $\mathcal{Q}$.

The following two corollaries are immediate.

Corollary 1.14. *Suppose that $\ell \in \mathcal{D}$, k and j positive integers. Then there are infinitely many non-nested arithmetic progressions $An + B$ such that for every $0 \leq m \leq \ell^j - 1$,*

$$\mathcal{R}(m, \ell^j, An + B) \equiv 0 \pmod{\ell^k}.$$

Corollary 1.15. *Suppose that $\ell \in \mathcal{D}$, and k a positive integer. Then there are infinitely many non-nested arithmetic progressions $An + B$ such that the statistic provides a proof of the congruence*

$$\Delta_1(An + B) \equiv 0 \pmod{\ell^k}.$$

In Section 2, in the spirit of the dissections for the generating functions of the crank and rank found in Ramanujan's Lost Notebook, as shown in [G], we prove Theorem 1.12.

In Section 3, we provide preliminaries on modular forms and Klein forms needed in the proofs of Theorems 1.8 and 1.13. These proofs are found in Sections 4 and 5, respectively. The proofs of two technical propositions are found in Section 6.

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2. The Statistic and its Generating Function

We begin by stating the closed forms of the generating functions that specialize to Theorems 1.2 and 1.4, respectively.

Theorem 2.1. *Let $X_0 := 1$, and $X_m := x_1 x_2 \cdots x_m$, $(m \geq 1)$. For $n \geq 1$,*

$$h_n(x_1, x_2, \ldots, x_{3n+1}) = \prod_{j=1}^{3n+1} \frac{1}{1 - X_j} \times \prod_{i=0}^{n-1} \frac{1 - X_{3i+1}X_{3i+3}}{1 - X_{3i+1}x_{3i+3}}.$$

Theorem 2.2. *Let $Y_1 := 1$ and $Y_n := y_2 y_3 \ldots y_n$. For $n \geq 2$,*

$$h_n^*(y_2, y_3, \ldots, y_{3n+1}) = \prod_{j=2}^{3n+1} \frac{1}{1 - Y_j} \times \prod_{i=0}^{n-1} \frac{1 - Y_{3i+1}Y_{3i+3}}{1 - Y_{3i+1}y_{3i+3}}.$$

These generating functions are now used to establish a generating function for the statistic:

Theorem 2.3. *We have the following generating function for the statistic $\mathcal{R}(m,n)$:*

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\mathcal{R}(m,n)z^mq^n=\prod_{n\geq 0}\frac{(1-zq^{6n+4})(1-z^{-1}q^{6n+2})}{(1-zq^{n+1})(1-q^{3n+2})(1-z^{-1}q^{n+1})(1-q^{3n+1})}.$$

Proof of Theorem 2.3. To explain the first set of brackets in Definition 1.11, it suffices to examine the following factor from Theorem 2.1:

$$\begin{aligned}
&\frac{1-zX_{3i+1}X_{3i+3}}{(1-zX_{3i+2})(1-X_{3i+1}x_{3i+3})}\\
&=\sum_{k=0}^{\infty}(zX_{3i+1}x_{3i+2})^k\cdot\frac{1-zX_{3i+1}^2x_{3i+2}x_{3i+3}}{1-X_{3i+1}x_{3i+3}}\\
&=\frac{1}{1-X_{3i+1}x_{3i+3}}\Big\{\sum_{k=0}^{\infty}z^kX_{3i+1}^kx_{3i+2}^k-\sum_{k=0}^{\infty}z^{k+1}X_{3i+1}^{k+2}x_{3i+2}^{k+1}x_{3i+3}\Big\}\\
&=\frac{1}{1-X_{3i+1}x_{3i+3}}\Big\{1+\sum_{k=1}^{\infty}z^kX_{3i+1}^kx_{3i+2}^k(1-X_{3i+1}x_{3i+3})\Big\}\\
&=\sum_{k=0}^{\infty}X_{3i+1}^kx_{3i+3}^k+\sum_{k=1}^{\infty}z^kX_{3i+2}^k.
\end{aligned}$$

In other words, the z is effectively counting the contribution to the largest part in terms of $1/(1-zX_k)$, while ignoring the contribution from $1/(1-X_{3i+1}x_{3i+3})$. In a similar fashion, the analogous factor from Theorem 2.2 explains the second set of brackets in Definition 1.11. The generating function for the statistic follows. □

We define

$$F(z,q):=\frac{(zq^4;q^6)_\infty(z^{-1}q^2;q^6)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty(q^2;q^3)_\infty(q;q^3)_\infty},\tag{2.1}$$

where this is the right hand side of the equation in Theorem 2.3, and we use Theorem 2.3 to obtain the following dissection.

Theorem 2.4. *Let $\rho=e^{2\pi i/3}$, then*

$$F(\rho,q)=A_{4,4}(q^6)-\rho q^4A_{4,1}(q^6)-\rho^2q^2A_{4,2}(q^6),$$

where

$$A_{k,i}(q):=\prod_{1\leq n\not\equiv 0,\pm i\pmod{2k+1}}\frac{1}{1-q^n}.$$

We recall the notation for Ramanujan's Theta function (p. 11, [AB]):

$$f(a,b):=\sum_{n\in\mathbb{Z}}a^{n(n+1)/2}b^{n(n-1)/2}=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.\tag{2.2}$$

A simple specialization gives,

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{an^2+bn} = (q^{2a};q^{2a})_\infty(q^{-b+a};q^{2a})_\infty(q^{b+a};q^{2a})_\infty. \tag{2.3}$$

The following two lemmas prove the above theorem.

Lemma 2.5. *Let* $\rho = e^{2\pi i/3}$. *Then* $F(\rho,q) = (\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty$.

Proof. This follows from a series of basic q-series manipulations.

$$\begin{aligned}F(\rho,q) &= \frac{(\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty}{(\rho q;q)_\infty(\rho^{-1}q;q)_\infty(q^2;q^3)_\infty(q;q^3)_\infty}\\ &= \frac{(\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty}{(\rho q;q)_\infty(\rho^{-1}q;q)_\infty(q^2;q^3)_\infty(q;q^3)_\infty}\cdot\frac{(q;q)_\infty}{(q;q)_\infty}\\ &= \frac{(\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty(q;q)_\infty}{(q^3;q^3)_\infty(q^2;q^3)_\infty(q;q^3)_\infty} = (\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty.\end{aligned}$$

□

Lemma 2.6. *Let* $\rho = e^{2\pi i/3}$. *Then*

$$(\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty = A_{4,4}(q^6) + \rho q^4 A_{4,1}(q^6) + \rho^2 q^2 A_{4,2}(q^6).$$

Proof. With Ramanujan's Theta function notation, we have

$$\begin{aligned}(\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty &= (\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty\frac{(q^6;q^6)_\infty}{(q^6;q^6)_\infty}\\ &= \frac{1}{(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-\rho q^4)^{n(n+1)/2}(-\rho^{-1}q^2)^{n(n-1)/2}\\ &= \frac{1}{(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\rho^n q^{3n^2+n} = \frac{1}{(q^6;q^6)_\infty}\sum_{k=0}^{2}\rho^k\sum_{n=-\infty}^{\infty}(-1)^{3n+k}q^{(3n+k)(9n+3k+1)}.\end{aligned}$$

Substituting in the values of k, we obtain

$$\begin{aligned}(\rho q^4;q^6)_\infty(\rho^{-1}q^2;q^6)_\infty =& \frac{1}{(q^6;q^6)_\infty}\Big\{\sum_{n=-\infty}^{\infty}(-1)^n q^{3n(9n+1)}\\ &- \rho\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n+1)(9n+4)} + \rho^2\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n+2)(9n+7)}\Big\}.\end{aligned}$$

With (2.3), and more calculation, we obtain the desired result. □

Proof of Theorem 1.12. From Theorem 2.4 it is immediate that

$$\sum_{m=-\infty}^{\infty}\mathcal{R}(m,2n+1)\rho^m = \sum_{k=0}^{2}\mathcal{R}(k,3,2n+1)\rho^k = 0,$$

since $F(z,q)$ is supported only on even powers of q. The left-hand side is polynomial in $\rho = e^{2\pi i/3}$ over $\mathbb{Z}$. Because the minimal polynomial for ρ over $\mathbb{Q}$ is $p(x) = 1+x+x^2$, it follows that

$$\mathcal{R}(0,3,2n+1) = \mathcal{R}(1,3,2n+1) = \mathcal{R}(2,3,2n+1).$$

□

3. Preliminaries for the Proofs of Theorems 1.8 and 1.13

3.1. **Modular Forms Background.** The reader can refer to [K] and [O] for basic facts about modular forms. We note that there are several ways to produce new modular forms from old ones. Among them are taking a twist, applying the Hecke Operator, and restricting the q-expansion to terms that lie in certain arithmetic progressions.

Definition 3.1. Suppose that $f(\tau)=\sum_{n=0}^{\infty}a(n)q^n \in M_k(\Gamma_0(N),\chi)$, and that ψ is a Dirichlet character. Then the *twist* of f by ψ is

$$(f\otimes\psi)(\tau):=\sum_{n\geq 0}\psi(n)a(n)q^n. \tag{3.1}$$

Definition 3.2. Suppose that $f(\tau)=\sum_{n=0}^{\infty}a(n)q^n \in M_k(\Gamma_0(N),\chi)$, and $p\nmid N$ prime. Then the action of the p-th Hecke Operator $T_p^{k,\chi}:=T(p)$ is given by

$$f(\tau)\mid T(p):=\sum_{n\geq 0}(a(pn)+\chi(p)p^{k-1}a(n/p))q^n. \tag{3.2}$$

We also have the following basic properties:

Proposition 3.3. *Suppose that $f(\tau)\in M_k(\Gamma_0(N),\chi)$, and $p\nmid N$ prime. Then the action of $T(p)$ is space-preserving, i.e.,*

$$f(\tau)\mid T(p)\in M_k(\Gamma_0(N),\chi).$$

If ψ is a character with modulus M, then

$$(f\otimes\psi)(\tau)\in M_k(\Gamma_0(NM^2),\chi\psi^2).$$

We note that the twist of a modular form by a quadratic character can be written in terms of the slash operator. (We surpress the weight term in the slash operator as the expression is independent of it). If p is a prime, where the Gauss sum $g:=g_p=\sum_{v=1}^{p-1}(\frac{v}{p})e^{2\pi iv/p}$, we can write (see p. 128 [K])

$$f(\tau)\otimes\left(\frac{\cdot}{p}\right)=\frac{g}{p}\sum_{v=1}^{p-1}\left(\frac{v}{p}\right)f(\tau)\mid\begin{pmatrix}1 & -v/p\\ 0 & 1\end{pmatrix}. \tag{3.3}$$

Discarding coefficients that lie in certain arithmetic progressions also produces modular forms.

Proposition 3.4. *Suppose that $f(\tau)=\sum_{n=1}^{\infty}a(n)q^n\in S_k(\Gamma_1(N))$, where k is an integer. If $t\geq 1$ and $0\leq r\leq t-1$, then*

$$\sum_{n\equiv r \pmod t}a(n)q^n\in S_k(\Gamma_1(Nt^2)).$$

In Section 5 we will need to simultaneously find congruences for modular forms of differing weights and levels. The following theorem is based on modifictions by Ahlgren and Ono [AO], [O2], and Mahlburg [M] to a classical result due to Deligne and Serre on Galois representations associated to modular forms.

Theorem 3.5. *Suppose that k_i and N_i are integers, and that χ_i is a Dirichlet character for $1 \leq i \leq r$. Let $g_1(\tau), \ldots, g_r(\tau)$ be integer weight modular forms with algebraic coefficients such that $g_i(\tau) \in M_{k_i}(\Gamma_0(N_i), \chi_i)$. If $M \geq 1$, then a positive proportion of primes $p \equiv -1 \pmod{N_1 \cdots N_r M}$ have the property that for every i,*

$$g_i(\tau) \mid T(p) \equiv 0 \pmod{M}.$$

3.2. **Siegel and Klein Forms.** Kubert and Lang [KL] studied transformation properties for certain modular functions which generalize $\eta^2(\tau)$. These generalizations, known as Klein and Siegel forms, are vital in studying the generating function of our statistic. Here we write

$$q = q_\tau = e^{2\pi i \tau}, \; q_z = e^{2\pi i z}, \text{ and } z = a_1\tau + a_2. \tag{3.4}$$

Definition 3.6. Let $(a_1, a_2) \in \mathbb{R}^2$,

(1) The (a_1, a_2)-*Siegel function* has the q-expansion

$$g_{(a_1,a_2)}(\tau) := -q_\tau^{(1/2)\mathbb{B}_2(a_1)} e^{2\pi i a_2(a_1-1)/2}(1 - q_z) \prod_{n=1}^{\infty} (1 - q_z q_\tau^n)(1 - q_z^{-1} q_\tau^n),$$

where $\mathbb{B}_2(X) = X^2 - X + \frac{1}{6}$ is the second Bernoulli polynomial.

(2) The (a_1, a_2)-*Klein form* is given by

$$t_{(a_1,a_2)}(\tau) := -\frac{i}{2\pi} \cdot \frac{g_{(a_1,a_2)}(\tau)}{\eta^2(\tau)}.$$

We have the following basic transformation properties (pp. 27-29 [KL]).

Proposition 3.7. *If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then*

$$t_{(a_1,a_2)}(\alpha\tau) = t_{(a_1,a_2)\alpha}(\tau).$$

Proposition 3.8. *Let $(a_1, a_2) \in \mathbb{R}^2$ and $(b_1, b_2) \in \mathbb{Z}^2$. Writing $(a_1 + b_1, a_2 + b_2) = (a_1, a_1) + (b_1, b_2)$, we have*

$$t_{(a_1+b_1,a_2+b_2)}(\tau) = t_{(a_1,a_2)+(b_1,b_2)}(\tau) = \varepsilon((a_1, a_2), (b_1, b_2)) t_{(a_1,a_2)}(\tau),$$

where $\varepsilon((a_1, a_2), (b_1, b_2))$ has absolute value 1, and is given explicitly by

$$\varepsilon((a_1, a_2), (b_1, b_2)) = (-1)^{b_1 b_2 + b_1 + b_2} e^{2\pi i (b_2 a_1 - b_1 a_2)}.$$

Letting the overbar denote reduction $\pmod{N}$, the above two propositions show,

Corollary 3.9. *If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $0 \leq s \leq N - 1$, then*

$$t_{(0,s/N)}(\tau)\Big|_{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \beta \cdot t_{(0,\overline{ds}/N)}(\tau),$$

where

$$\beta := e^{\left(\frac{cs + (ds - \overline{ds})}{2N} + \frac{cs(ds - \overline{ds}) - cs \cdot \overline{ds}}{2N^2}\right)}.$$

The following corollary describes a general case in which the multiplier β is trivial. Here $M^!$ denotes the space of weakly holomorphic forms.

Corollary 3.10. *Let $0 \leq r, s \leq N - 1$, then $t_{(r/N,s/N)}(\tau) \in M^!_{-1}(\Gamma_1(2N^2))$.*

4. Proof of Theorem 1.8

The proof borrows techniques from [AO]. We first establish a key theorem, in that we construct cusp forms whose coefficients capture the relevant values of $\Delta_1(n)$. We recall the defintions of $\mathcal{D}$, ϵ_ℓ, δ_ℓ and S_ℓ in (1.3), (1.4), and (1.5).

Theorem 4.1. *Suppose $\ell \in \mathcal{D}$ and that m is a positive integer. If $\beta \in S_\ell$, then there is an integer $\lambda_{\ell,m}$ and a modular form $F_{\ell,m,\beta}(\tau) \in S_{\lambda_{\ell,m}}(\Gamma_1(108\ell^5)) \cap \mathbb{Z}[[q]]$ such that*

$$F_{\ell,m,\beta}(\tau) \equiv \sum_{n=0}^{\infty} \Delta_1(\ell n + \beta) q^{18\ell n + 18\beta - 3} \pmod{\ell^m}.$$

The cusp form will be the product of two modular forms, one vanishing at all cusps a/c where $\ell^3 \nmid c$, the other vanishing at all a/c where $\ell^3 \mid c$. The latter is where we need $\ell \in \mathcal{D}$. The following useful proposition can be shown via basic facts about modular forms.

Proposition 4.2. *Let $\ell \geq 5$ be a prime. We define $E_{\ell,t}(\tau) := \frac{\eta^{\ell^t}(\tau)}{\eta(\ell^t \tau)}$.*

1. $E_{\ell,t}(\tau) \in M_{\frac{\ell^t-1}{2}}(\Gamma_0(\ell^t), \chi_{\ell,t})$, *where* $\chi_{\ell,t}(\cdot) := \left(\frac{(-1)^{\frac{\ell^t-1}{2}} \ell^t}{\cdot}\right)$.
2. *If $\ell \nmid a$, $0 \leq b < t$, then $ord_{a/\ell^b}(E_{\ell,t}) > 0$, i.e. $E_{\ell,t}(\tau)$ vanishes at those cusps of $\Gamma_0(\ell^t)$ not equivalent to ∞.*
3. $E_{\ell,t}(\tau)^{\ell^{m-1}} \equiv 1 \pmod{\ell^m}$.

Define

$$\frac{1}{g(\tau)} := \frac{\eta(2\tau)\eta(3\tau)}{\eta(\tau)^3 \eta(6\tau)} = \sum_{n=0}^{\infty} \Delta_1(n) q^{n-\frac{1}{6}}, \tag{4.1}$$

If $\ell \geq 5$ is prime, one can show

$$f_\ell(\tau) = \sum_{n=1}^{\infty} a_\ell(n) q^n := \frac{g^\ell(\ell\tau)}{g(\tau)} \in M_{\ell-1}(\Gamma_0(6\ell), \chi_{triv}). \tag{4.2}$$

This implies

$$\sum_{n=1}^{\infty} a_\ell(n) q^n = \Big(\sum_{n=0}^{\infty} \Delta_1(n) q^{n+\delta_\ell}\Big) \cdot \prod_{n=1}^{\infty} \frac{(1-q^{\ell n})^{3\ell}(1-q^{6\ell n})^\ell}{(1-q^{2\ell n})^\ell (1-q^{3\ell n})^\ell}. \tag{4.3}$$

Further, we can show

$$\tilde{f}_\ell(\tau) := f_\ell(\tau) - \varepsilon_\ell \cdot f_\ell \otimes \left(\tfrac{\cdot}{\ell}\right)(\tau) = \sum_{n=1}^{\infty} \left(1 - \varepsilon_\ell \cdot \left(\tfrac{n}{\ell}\right)\right) a_\ell(n) q^n \in M_{\ell-1}(\Gamma_0(6\ell^3), \chi_{triv}). \tag{4.4}$$

We want to show that the quotient $\tilde{f}_\ell(\tau)/g^\ell(\ell\tau)$ vanishes at all cusps a/c where $\ell^3 \mid c$, $\ell \in \mathcal{D}$. We subdivide those cusps into four groups and define v_i to be the order of vanishing of $f_\ell(\tau)$ at any cusp of $a/c \in \mathcal{C}_i$, and v_i' that of $g^\ell(\ell\tau)$.

1. $\mathcal{C}_1 := \{\frac{a}{c} : 6\ell^3 \mid c\}$, $v_1 = (\frac{\ell^2-1}{24} \cdot 4)$, $v_1' = (\frac{\ell^2}{24} \cdot 4)$
2. $\mathcal{C}_2 := \{\frac{a}{c} : 3\ell^3 \mid c,\ 2 \nmid c\}$, $v_2 = (\frac{\ell^2-1}{24} \cdot 1)$, $v_2' = (\frac{\ell^2}{24} \cdot 1)$,
3. $\mathcal{C}_3 := \{\frac{a}{c} : 2\ell^3 \mid c,\ 3 \nmid c\}$, $v_3 = (\frac{\ell^2-1}{24} \cdot \frac{4}{3})$, $v_3' = (\frac{\ell^2}{24} \cdot \frac{4}{3})$,

(4) $\mathcal{C}_4 := \{\frac{a}{c} : \ell^3 \mid c;\ 2,\ 3 \nmid c\},\ v_4 = (\frac{\ell^2-1}{24} \cdot \frac{7}{3}),\ v_4' = (\frac{\ell^2}{24} \cdot \frac{7}{3})$.

Next we have the following proposition whose proof we defer to Section 6.

Proposition 4.3. *Given $\frac{a}{c} \in \mathcal{C}_i$, $1 \le i \le 4$, choose b, d such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, and define α by $f_\ell \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \cdot q^{v_i} + \dots$, then*

$$f_\ell \otimes \left(\frac{\cdot}{\ell}\right) \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{v_i}{\ell}\right) \cdot \alpha \cdot q^{v_i} + \dots .$$

For $\ell \in \mathcal{D}$, quadratic reciprocity gives us,

$$\varepsilon_\ell = \left(\frac{v_1}{\ell}\right) = \left(\frac{v_2}{\ell}\right) = \left(\frac{v_3}{\ell}\right) = \left(\frac{v_4}{\ell}\right). \tag{4.5}$$

We note that for $a/c \in \mathcal{C}_i$, and b, d chosen such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$\frac{1}{g^\ell(\ell\tau)} \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = q^{-v_i'} + \dots . \tag{4.6}$$

Equation (4.4) and Proposition 4.3 yield,

Proposition 4.4. *Given $\ell \in \mathcal{D}$ and a cusp a/c such that $\ell^3 \mid c$,*

$$ord_{a/c}\Big(\frac{\tilde{f}_\ell(\tau)}{g^\ell(\ell\tau)}\Big) \ge v_i + 1 - v_i' > 0. \tag{4.7}$$

Proof of Theorem 4.1. We consider $\frac{\tilde{f}_\ell(\tau)}{g^\ell(\ell\tau)} \cdot E_{\ell,3}(\tau)^{\ell^{m'}}$. By Propositions 4.2(2) and 4.4, with m' sufficiently large, this vanishes at all the cusps. By (4.3), (4.4) and Propositions 4.2(3), 4.4, we have

$$\frac{\tilde{f}_\ell(\tau)}{g^\ell(\ell\tau)} \cdot E_{\ell,3}(\tau)^{\ell^{m'}} \equiv \sum_{n \equiv 0 \pmod{\ell}} \Delta_1(n - \delta_\ell) q^{n - \frac{\ell^2}{6}} + 2 \cdot \sum_{(\frac{n}{\ell}) = -\varepsilon_\ell} \Delta_1(n - \delta_\ell) q^{n - \frac{\ell^2}{6}} \pmod{\ell^m}. \tag{4.8}$$

It follows that $\frac{\tilde{f}_\ell(18\tau)}{g^\ell(\ell 18\tau)} \cdot E_{\ell,3}(18\tau)^{\ell^{m'}}$ is a cusp form on $\Gamma_0(18 \cdot 6\ell^3)$, and we then apply Proposition 3.4. □

Proof of Theorem 1.8. Fix $\ell \in \mathcal{D}$, an integer $\beta \in S_\ell$, and write

$$F_{\ell,m,\beta}(\tau) = \sum_{n=1}^{\infty} a_{\ell,m,\beta}(n) q^n \equiv \sum_{n \equiv 18\beta - 3 \pmod{18\ell}} \Delta_1\left(\frac{n+3}{18}\right) q^n \pmod{\ell^m}. \tag{4.9}$$

By Theorem 3.5, for a positive proportion of primes $\mathcal{Q} \equiv -1 \pmod{18\ell}$,

$$F_{\ell,m,\beta}(\tau) \mid T(\mathcal{Q}) \equiv 0 \pmod{\ell^m}. \tag{4.10}$$

Theorem 1.8 then follows from the following. We see that if $n \equiv 3 - 18\beta \pmod{18\ell}$, that $\mathcal{Q}n \equiv 18\beta - 3 \pmod{18\ell}$; and if $\gcd(\mathcal{Q}, n) = 1$, we can use (3.2) to obtain

$$0 \equiv a_{\ell,m,\beta}(\mathcal{Q}n) \equiv \Delta_1\left(\frac{\mathcal{Q}n+3}{18}\right) \pmod{\ell^m}. \tag{4.11}$$

□

5. Proof of Theorem 1.13

We borrow techniques from [AO] and [M], and begin by producing a generating function for $\mathcal{R}(m, N, n)$ in terms of Klein forms. We again remind ourselves of the defintions of $\mathcal{D}$, ϵ_ℓ, δ_ℓ and S_ℓ in (1.3), (1.4), and (1.5). Recalling (2.1), setting $N := \ell^j$, $\zeta := e^{2\pi i/N}$, λ a partition, and letting $0 \le s \le N-1$, we examine the sum

$$\begin{aligned}\frac{1}{N}\sum_{s=0}^{N-1} F(\zeta^s, z)\zeta^{-ms} &= \frac{1}{N}\sum_{s=0}^{N-1}\sum_{\lambda}\zeta^{\mathcal{R}(\lambda)\cdot s - ms}q^{|\lambda|} \\ &= \sum_{\lambda} q^{|\lambda|}\left(\frac{1}{N}\zeta^{s(\mathcal{R}(\lambda)-m)}\right) = \sum_{n\ge 0}\mathcal{R}(m,N,n)q^n,\end{aligned} \tag{5.1}$$

which leads to

$$\begin{aligned}\sum_{n\ge 0}\mathcal{R}(m,N,n)q^n = \frac{1}{N}\sum_{s=1}^{N-1}&\frac{(\zeta^s q^4;q^6)_\infty(\zeta^{-s}q^2;q^6)_\infty\zeta^{-ms}}{(\zeta^s q;q)_\infty(\zeta^{-s}q;q)_\infty(q^2;q^3)_\infty(q;q^3)_\infty} \\ &+ \frac{1}{N}\sum_{n\ge 0}\Delta_1(n)q^n.\end{aligned} \tag{5.2}$$

We recall (4.1), Definition 3.6, and make the substitution $\zeta \to \zeta^{-1}$ to obtain

$$\sum_{n\ge 0}\mathcal{R}(m,N,n)q^n = \sum_{s=1}^{N-1}\frac{h_s q^{\frac{1}{6}}}{g(\tau)}\cdot\frac{t_{(1/3,s/N)}(6\tau)}{t_{(0,s/N)}(\tau)}\cdot\frac{\eta^3(6\tau)}{\eta(2\tau)}\cdot\frac{\zeta^{ms}}{N} + \frac{1}{N}\sum_{n\ge 0}\Delta_1(n)q^n, \tag{5.3}$$

where $h_s := -\zeta^{s/2}(1-\zeta^{-s})/\zeta^{-s/6}$.

Now we define a function more amenable to being analyzed by modular forms:

$$\begin{aligned}g_m(\tau) &:= \Big(\sum_{n\ge 0} N\cdot\mathcal{R}(m,N,n)q^{n+\delta_\ell}\Big)\prod_{n\ge 1}\frac{(1-q^{\ell n})^\ell(1-q^{6\ell n})^\ell}{(1-q^{2\ell n})^\ell(1-q^{3\ell n})^\ell} \\ &= \sum_{s=1}^{N-1}\frac{g^\ell(\ell\tau)}{g(\tau)}\cdot\frac{h_s t_{(1/3,s/N)}(6\tau)}{t_{(0,s/N)}(\tau)}\cdot\frac{\eta^3(6\tau)}{\eta(2\tau)} + \frac{g^\ell(\ell\tau)}{g(\tau)},\end{aligned} \tag{5.4}$$

with $P_m(\tau)$ and $P(\tau)$ denoting the two summands in the final expression.

As in the previous section, we proceed to relate $g_m(\tau)$ to cusp forms by first subdividing the cusps a/c, $\ell N|c$ into four groups.

(1) $\mathcal{C}_1' := \{\frac{a}{c} : 6\ell N \mid c\}$,
(2) $\mathcal{C}_2' := \{\frac{a}{c} : 3\ell N \mid c,\ 2\nmid c\}$,
(3) $\mathcal{C}_3' := \{\frac{a}{c} : 2\ell N \mid c,\ 3\nmid c\}$,
(4) $\mathcal{C}_4' := \{\frac{a}{c} : \ell N \mid c;\ 2,\ 3\nmid c\}$.

The following proposition will be proved in Section 6:

Proposition 5.1. *Given* $1 \le s \le N-1$, $\frac{a}{c} \in \mathcal{C}_i'$, *pick* b, d *with* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. *Define* α_s *by*

$$\frac{g^\ell(\ell\tau)}{g(\tau)} \frac{t_{(1/3,s/N)}(6\tau)}{t_{(0,s/N)}(\tau)} \frac{\eta^3(6\tau)}{\eta(2\tau)} \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha_s \cdot q^{v_i} + \dots \ .$$

Then

$$\frac{g^\ell(\ell\tau)}{g(\tau)} \frac{t_{(1/3,s/N)}(6z)}{t_{(0,s/N)}(\tau)} \frac{\eta^3(6\tau)}{\eta(2\tau)} \otimes \left(\frac{\cdot}{\ell}\right) \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{v_i}{\ell}\right) \cdot \alpha_s \cdot q^{v_i} + \dots \ ,$$

where the v_i*'s are as in the previous section.*

With this and previous arguments, we prove the following proposition, which provides us with a theorem crucial to the proof of Theorem 1.13. We recall the tilde notation from (4.4) for the following theorem and proposition.

Proposition 5.2. *Let* $\ell \in \mathcal{D}$. *If* k *is sufficiently large, there exist integers* λ, $\lambda' \ge 1$ *and some Dirichlet character* χ *such that*

(1) $\frac{\tilde{P}(18\tau)}{g^\ell(18\ell\tau)} \cdot E_{j+1}(18\tau)^{\ell^k} \in S_{\lambda'}(\Gamma_0(108\ell^{max\{3,j+1\}}), \chi)$

(2) $\frac{\tilde{P}_m(18\tau)}{g^\ell(18\ell\tau)} \cdot E_{j+1}(18\tau)^{\ell^k} \in S_\lambda(\Gamma_1(1944\ell^2 N^2))$.

Proof of Proposition 5.2. For (1), recall that $P(\tau) = \frac{g^\ell(\ell\tau)}{g(\tau)} \in M_{\ell-1}(\Gamma_0(6\ell), \chi_{triv})$ and then modify the arguments of the previous section.

For the (2), we know that $E_{j+1}(\tau)$ vanishes at each cusp a/c, $\ell N \nmid c$. Once k is taken to be sufficiently large, it only remains to show that $\tilde{P}_m(\tau)/g^\ell(\ell\tau)$ vanishes at each cusp a/c with $\ell N \mid c$, but this follows from Proposition 5.1, equation (4.5), and quadratic reciprocity. To determine the appropriate level of the congruence subgroup, basic facts about modular forms and Corollary 3.10 yield $P_m(3\tau) \in M_\ell^!(\Gamma_1(2^2 \cdot 3^4 \cdot N^2))$, and the rest is straightforward. ☐

Combining the above proposition with the definition of $g_m(\tau)$ in (5.4) and the congruence properties from Proposition 4.2, we obtain the following theorem. This is essentially heading in the same direction as that of Theorem 4.1; here, however, we work with two modular forms.

Theorem 5.3. *For* $k \ge 0$ *and* $0 \le m \le N-1$ *there is a character* χ, *positive integers* λ *and* λ', *and modular forms*

(1) $\mathcal{F}(\tau) \in S_{\lambda'}(\Gamma_0(108\ell^{max\{3,j+1\}}), \chi)$, *and*

(2) $\mathcal{F}_m(\tau) \in S_\lambda(\Gamma_1(1944\ell^2 N^2))$,

such that $\frac{\tilde{g}_m(18\tau)}{g^\ell(18\ell\tau)} \equiv \mathcal{F}_m(\tau) + \mathcal{F}(\tau) \pmod{\ell^k}$.

Proof of Theorem 1.13. By (4.4) and (5.4), we have

$$\frac{\tilde{g}_m(18\tau)}{g^\ell(18\ell\tau)} = \sum_{n\equiv 0 \pmod{\ell}} N \cdot \mathcal{R}(m,N,n-\delta_\ell) q^{18n-3\ell^2} + 2 \sum_{\left(\frac{n}{\ell}\right) = -\varepsilon_\ell} N \cdot \mathcal{R}(m,N,n-\delta_\ell) q^{18n-3\ell^2} \tag{5.5}$$

Restricting the above sum to those indices $n' \equiv \beta + \delta_\ell \pmod{\ell}$, $\beta \in S_\ell$, yields a new series where λ_β equals 1 if $\beta = -\delta_\ell$ and equals 2 otherwise.

$$\begin{aligned} g_{m,\beta}(\tau) :&= \lambda_\beta \cdot \sum_{n'=\beta+\delta_\ell+\ell n} N \cdot \mathcal{R}(m, N, n' - \delta_\ell) q^{18n'-3\ell^2} \\ &= \lambda_\beta \cdot \sum_{n \geq 0} N \cdot \mathcal{R}(m, N, \ell n + \beta) q^{18\beta+18\delta_\ell+18\ell n-3\ell^2} \\ &= \lambda_\beta \cdot \sum_{n \equiv 18\beta - 3 \pmod{18\ell}} N \cdot \mathcal{R}\left(m, N, \frac{n+3}{18}\right) q^n \end{aligned} \tag{5.6}$$

However, upon examining Theorem 5.3, we can restrict the q-expansions of $\mathcal{F}_m(\tau)$ and $\mathcal{F}(\tau)$ to those indices with $n' \equiv \beta + \delta_\ell \pmod{\ell}$ to obtain

$$g_{m,\beta}(\tau) \equiv \mathcal{F}_{m,\beta}(\tau) + \mathcal{F}_\beta(\tau) \pmod{\ell^k}, \tag{5.7}$$

and the following from Proposition 3.4:

$$\mathcal{F}_{m,\beta}(\tau) \in S_\lambda(\Gamma_1(1944\ell^4 N^2)), \tag{5.8}$$

$$\mathcal{F}_\beta(\tau) \in S_{\lambda'}(\Gamma_0(108\ell^{2+max\{3,j+1\}}), \chi). \tag{5.9}$$

Theorem 3.5 provides a positive proportion of primes $\mathcal{Q} \equiv -1 \pmod{18\ell}$ such that

$$\mathcal{F}_{m,\beta}(\tau) \mid T(\mathcal{Q}) \equiv \mathcal{F}_\beta(z) \mid T(\mathcal{Q}) \equiv 0 \pmod{\ell^k} \tag{5.10}$$

for all m. This in turn gives that $g_{m,\beta}(\tau) \mid T(\mathcal{Q}) \equiv 0 \pmod{\ell^k}$ for all m. Arguing as in the proof of Theorem 1.8 we obtain for all $n \equiv 3 - 18\beta \pmod{18\ell}$, $\gcd(n, \mathcal{Q}) = 1$ that

$$\lambda_\beta \cdot N \cdot \mathcal{R}\left(m, N, \frac{\mathcal{Q}n+3}{18}\right) \equiv 0 \pmod{\ell^k}.$$

Theorem 1.13 follows by noting that because N is fixed and k is arbitrary, dividing by N proves the congruences. □

6. Proofs of Propositions 4.4 and 5.1

6.1. **Proof of Proposition 4.4.** We restrict ourselves to the case $2\ell N \mid c$, $3 \nmid c$, the other three cases being similar. Let $\delta \in \mathbb{Z}^+$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and $(\cdot,\cdot)$ denote the greatest common divisor, then we can write

$$\begin{pmatrix} \delta a & \delta b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta a/(c,\delta) & \gamma_{\delta,b} \\ c/(c,\delta) & \gamma_{\delta,d} \end{pmatrix} \begin{pmatrix} (c,\delta) & B_\delta \\ 0 & \delta/(c,\delta) \end{pmatrix}, \tag{6.1}$$

$$\begin{pmatrix} \ell\delta a & \ell\delta b \\ c & d \end{pmatrix} = \begin{pmatrix} \delta a/(c,\delta) & \gamma'_{\delta,b} \\ c/\ell(c,\delta) & \gamma'_{\delta,d} \end{pmatrix} \begin{pmatrix} \ell(c,\delta) & \ell B_\delta \\ 0 & \delta/(c,\delta) \end{pmatrix}, \tag{6.2}$$

where the left matrix in each product is in $SL_2(\mathbb{Z})$,

$$\gamma'_{\delta,d} := \gamma_{\delta,d},\ \gamma'_{\delta,b} := \ell\gamma_{\delta,b}, \text{ and } B_\delta = \gamma_{\delta,d} \cdot \delta \cdot b - \gamma_{\delta,b} \cdot d. \tag{6.3}$$

The transformation formula for Dedekind's eta function (see p. 163 [R]) yields,

$$\frac{\eta^\ell(\ell\delta\tau)}{\eta(\delta\tau)} \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{\delta a/(c,\delta)}{\ell}\right) e^{\frac{2\pi i}{24}(\ell^2-1)\frac{(c,\delta)}{\delta}B_\delta} q^{\frac{1}{24}\frac{(c,\delta)^2}{\delta}(\ell^2-1)} + \dots\ . \tag{6.4}$$

It then follows

$$f_\ell(\tau)\Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha \cdot q^{ov} + \dots . \tag{6.5}$$

where

$$\begin{aligned} \alpha &:= e^{\frac{2\pi i}{24}(\ell^2-1)\left\{\frac{(c,1)}{1}3B_1 - \frac{(c,2)}{2}B_2 - \frac{(c,3)}{3}B_3 + \frac{(c,6)}{6}B_6\right\}}, \\ ov &:= \tfrac{1}{24}(\ell^2-1)\{\tfrac{(c,1)^2}{1}3 - \tfrac{(c,2)^2}{2} - \tfrac{(c,3)^2}{3} + \tfrac{(c,6)^2}{6}\}. \end{aligned} \tag{6.6}$$

We note $ov = v_3$. By recalling how $\gamma_{\delta,b}$, $\gamma_{\delta,d}$, and B_δ were defined,

$$\begin{pmatrix} 3a & 3b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a & \gamma_{3,b} \\ c & \gamma_{3,d} \end{pmatrix}\begin{pmatrix} 1 & B_3 \\ 0 & 3 \end{pmatrix}, \ \begin{pmatrix} 6a & 6b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a & \gamma_{6,b} \\ c/2 & \gamma_{6,d} \end{pmatrix}\begin{pmatrix} 2 & B_6 \\ 0 & 3 \end{pmatrix}, \tag{6.7}$$

and by setting $\gamma_{6,d} := \gamma_{3,d}$, and $\gamma_{6,b} := 2\gamma_{3,b}$, we can relate B_3 and B_6:

$$B_6 = \gamma_{6,d} \cdot 6 \cdot b - \gamma_{6,b} \cdot d = 2 \cdot B_3. \tag{6.8}$$

With this and the fact that $24 \mid \ell^2 - 1$, the expression for α simplifies to

$$\alpha = e^{\frac{2\pi i}{24}(\ell^2-1)\left\{\frac{1}{3}B_3\right\}} = e^{\frac{2\pi i}{24}(\ell^2-1)\left\{\frac{-\gamma_{3,b}}{3}d\right\}}. \tag{6.9}$$

We recall expression (3.3), define $\gamma_\nu := \begin{pmatrix} 1 & -\nu/\ell \\ 0 & 1 \end{pmatrix}$, and choose $\nu' \equiv d^2\nu \pmod{\ell}$:

$$\begin{aligned} f_\ell \otimes (\tfrac{\cdot}{\ell})(\tau)\Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\nu}{\ell}\right) f_\ell \Big| \Big[\gamma_\nu\Big] \Big| \Big[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Big] \\ &= \frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\nu}{\ell}\right) f_\ell \Big| \Big[\gamma_\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} {\gamma_{\nu'}}^{-1}\gamma_{\nu'}\Big] \\ &= \frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\nu}{\ell}\right) f_\ell \Big| \Big[\gamma_\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} {\gamma_{\nu'}}^{-1}\Big] \Big| \Big[\gamma_{\nu'}\Big]. \end{aligned} \tag{6.10}$$

By choice of ν',

$$\gamma_\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} {\gamma_{\nu'}}^{-1} = \begin{pmatrix} a - c\nu/\ell & b + (\nu' a - \nu d)/\ell - c\nu\nu'/\ell^2 \\ c & d + c\nu'/\ell \end{pmatrix} =: \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \tag{6.11}$$

and thus define $a',\ b',\ c',\ d'$. Using (6.5) and (6.9),

$$f_\ell \otimes (\tfrac{\cdot}{\ell})(\tau)\Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\nu}{\ell}\right) e^{\frac{2\pi i}{24}(\ell^2-1)\left\{\frac{-\gamma'_{3,b}\cdot d'}{3} - \frac{4}{3}\frac{(\nu')}{\ell}\right\}} q^{v_3} + \dots . \tag{6.12}$$

Considering the analog of (6.1) for (6.11), one sees that $\gamma'_{3,b} \equiv \gamma_{3,b} \pmod 3$. Coupled with the definition $d' := d + c\nu'/\ell$ we have

$$\begin{aligned} f_\ell \otimes (\tfrac{\cdot}{\ell})(\tau)\Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\Big(\frac{\nu}{\ell}\Big) e^{\frac{2\pi i}{24}(\ell^2-1)\{\frac{-\gamma_{3,b}\cdot d}{3}\}} e^{\frac{2\pi i}{24}(\ell^2-1)\{\frac{-\gamma_{3,b}\cdot c}{3}-\frac{4}{3}\}\frac{\nu'}{\ell}} q^{v_3} + \dots \\ &= \frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\Big(\frac{\nu}{\ell}\Big)\cdot \alpha \cdot e^{\frac{2\pi i}{24}(\ell^2-1)\{\frac{-\gamma_{3,b}\cdot c}{3}-\frac{4}{3}\}\frac{\nu'}{\ell}} q^{v_3} + \dots\ . \end{aligned} \tag{6.13}$$

Recalling (6.1), we note $3a\cdot\gamma_{3,d} - \gamma_{3,b}\cdot c = 1$, and therefore $-\gamma_{3,b}\cdot c \equiv 1 \pmod 3$. From this it is clear that $\frac{-\gamma_{3,b}\cdot c}{3} - \frac{4}{3} \in \mathbb{Z}$ and that

$$\begin{aligned} f_\ell \otimes (\tfrac{\cdot}{\ell})(\tau)\Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{g}{\ell}\cdot g \cdot \Big(\frac{\frac{2\pi i}{24}(\ell^2-1)\{\frac{-\gamma_{3,b}\cdot c}{3} - \frac{4}{3}\}}{\ell}\Big)\cdot \alpha q^{v_3} + \dots \\ &= \frac{g^2}{\ell}\Big(\frac{-1}{\ell}\Big)\Big(\frac{v_3}{\ell}\Big)\cdot \alpha q^{v_3} + \dots = \Big(\frac{v_3}{\ell}\Big)\cdot \alpha q^{v_3} + \dots\ . \end{aligned} \tag{6.14}$$

6.2. **Proof of Proposition 5.1.** We examine the case $2\ell N \mid c,\ 3 \nmid c$. For the sake of exposition we further refine it to the subcase $4 \mid c$, with the other cases being similar. We compute the lead term for each of $P_{m,s}(\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $P_{m,s}\otimes(\frac{\cdot}{\ell})(\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where from Section 5,

$$P_{m,s}(\tau) := \frac{g^\ell(\ell\tau)}{g(\tau)}\frac{h_s t_{(1/3,s/N)}(6\tau)}{t_{(0,s/N)}(\tau)}\frac{\eta^3(6\tau)}{\eta(2\tau)}. \tag{6.15}$$

We reduce the problem to finding the lead terms for each of the expansions:

$$\frac{g^\ell(\ell\tau)}{g(\tau)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix}, t_{(1/3,s/N)}(6\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \frac{1}{t_{(0,s/N)}(\tau)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \text{and}\ \frac{\eta^3(6\tau)}{\eta(2\tau)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From the previous subsection:

$$\frac{g^\ell(\ell\tau)}{g(\tau)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{\frac{2\pi i}{24}(\ell^2-1)(\frac{-\gamma_{3,b}\cdot d}{3})}\cdot q^{\frac{1}{24}(\ell^2-1)\frac{4}{3}} + \dots\ . \tag{6.16}$$

By Corollary 3.9,

$$\frac{1}{t_{(0,s/N)}(\tau)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\beta_s t_{(0,\overline{ds}/N)}(\tau)} = \frac{1}{\beta_s(\frac{-i}{2\pi})\omega_{\overline{ds}}}\cdot q^0 + \dots\ . \tag{6.17}$$

Using the transformation law for Dedekind's eta function,

$$\frac{\eta^3(6\tau)}{\eta(2\tau)}\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Big(\frac{c/2}{3}\Big)e^{\frac{\pi i}{12}\{4ac\}}\cdot q^0 + \dots\ . \tag{6.18}$$

For the Klein form, we require more work. Note,

$$\begin{pmatrix} 6a & 6b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a & \gamma_{6,b} \\ c/2 & \gamma_{6,d} \end{pmatrix}\begin{pmatrix} 2 & B_6 \\ 0 & 3 \end{pmatrix}\ , \tag{6.19}$$

where the left matrix in the product is in $SL_2(\mathbb{Z})$. By Propositions 3.7 and 3.8,

$$t_{(1/3,s/N)}(6\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = t_{(1/3,s/N)}\left(\begin{pmatrix} 3a & \gamma_{6,b} \\ c/2 & \gamma_{6,d} \end{pmatrix}(\tfrac{2}{3}\tau + B_6)\right) \tag{6.20}$$
$$= t_{\left(a+\frac{cs}{2N}, \frac{\gamma_{6,b}}{3}+\frac{\gamma_{6,d}\cdot s}{N}\right)}(\tfrac{2}{3}\tau + B_6) = t_{(a_1,a_2)+(b_1,b_2)}(\tfrac{2}{3}\tau + B_6)$$
$$= \varepsilon((a_1,a_2),(b_1,b_2))t_{(a_1,a_2)}(\tfrac{2}{3}\tau + B_6),$$

where

$$(a_1,a_2) := \left(0, \tfrac{\overline{\gamma_{6,b}}}{3} + \tfrac{\overline{\gamma_{6,d}\cdot s}}{N}\right),\ (b_1,b_2) := \left(a + \tfrac{cs}{2N}, \tfrac{\gamma_{6,b}-\overline{\gamma_{6,b}}}{3} + \tfrac{\gamma_{6,d}\cdot s - \overline{\gamma_{6,d}\cdot s}}{N}\right), \tag{6.21}$$

and the overbars mean $\pmod 3$ and $\pmod N$ respectively. By Proposition 3.8,

$$t_{(1/3,s/N)}(6\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{\pi i(b_1b_2+b_1+b_2-b_1a_2-a_2)}(1-a^{2\pi i a_2})\cdot q^0 + \dots . \tag{6.22}$$

Piecing together the four lead terms and including h_s,

$$\alpha_s := h_s \cdot e^{2\pi i\left(\frac{\ell^2-1}{24}\right)\left(\frac{-\gamma_{3,b}\cdot d}{3}\right)} \cdot \frac{1}{\beta_s\cdot(\frac{-i}{2\pi})\cdot\omega_{\overline{ds}}} \cdot \left(\frac{c/2}{3}\right) e^{\frac{\pi i}{12}\{4ac\}} \tag{6.23}$$
$$\cdot\, e^{\pi i(b_1b_2+b_1+b_2-b_1a_2-a_2)}\cdot(1-e^{2\pi i a_2}),$$

so

$$P_{m,s}(\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha_s(a,b,c,d)\cdot q^{v_3} + \dots \tag{6.24}$$

We consider the expansion of the twisted term. We recall the defintions of a', b', c', d' in (6.11) and note that $a'_1 = a_1, a'_2 = a_2$. After much calculation we obtain

$$\alpha'_s = \alpha_s \frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\left(\frac{\nu}{\ell}\right)\cdot e^{\frac{2\pi i}{24}(\ell^2-1)\left\{\frac{-\gamma_{3,b}\cdot c-4}{3}\right\}\frac{\nu'}{\ell}}\cdot e^{\frac{\pi i}{12}\{-4c^2\frac{\nu}{\ell}\}} \tag{6.25}$$
$$\cdot\, e^{\pi i(b'_1b'_2-b_1b_2+b'_1-b_1+b'_2-b_2-a_2(b'_1-b_1))},$$

so

$$P_{m,s}\otimes(\tfrac{\cdot}{\ell})(\tau)\Big|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha_s\cdot\left\{\frac{g}{\ell}\sum_{\nu=0}^{\ell-1}\left[\left(\frac{\nu}{\ell}\right)\cdot e^{\frac{2\pi i}{24}(\ell^2-1)\left\{\frac{-\gamma_{3,b}\cdot c-4}{3}\right\}\frac{\nu'}{\ell}}\right.\right. \tag{6.26}$$
$$\left.\left.\cdot\, e^{2\pi i\{-\frac{c^2\nu}{6\ell}+\frac{1}{2}(b'_1b'_2-b_1b_2+b'_1-b_1+b'_2-b_2-a_2(b'_1-b_1))\}}q^{v_3}\right]\right\}.$$

To demonstrate that the second exponential is in fact 1, we first note that

$$b'_1 = b_1 - c\nu/\ell \text{ and } b'_2 = b_2 + \tfrac{\gamma_{6,b}-\overline{\gamma_{6,b}}}{3} + \tfrac{\gamma_{6,d}\cdot s-\overline{\gamma_{6,d}\cdot s}}{N}. \tag{6.27}$$

Noting that a is odd and $4\ell N \mid c$,

$$e^{2\pi i\left\{-\frac{c^2\nu}{6\ell}+\frac{1}{2}(b'_1b'_2-b_1b_2+b'_1-b_1+b'_2-b_2-a_2(b'_1-b_1))\right\}} = e^{2\pi i\left\{-\frac{c^2\nu}{6\ell}+\frac{1}{2}\left\{\left(a+\frac{cs}{2N}+1\right)\left(b'_2-b_2\right)+a_2c\frac{\nu}{\ell}\right\}\right\}}$$
$$= e^{2\pi i\left\{-\frac{c^2\nu}{6\ell}+\frac{c}{2}\left\{a_2\frac{\nu}{\ell}\right\}\right\}} = e^{2\pi i\left\{-\frac{c^2\nu}{6\ell}+\frac{c}{2}\left\{\left(\frac{\overline{\gamma_{6,b}}}{3}+\frac{\overline{\gamma_{6,d}\cdot s}}{N}\right)\frac{\nu}{\ell}\right\}\right\}} = e^{2\pi i\left\{\frac{c}{2}\left(\frac{\overline{\gamma_{6,b}}-c}{3}\right)\frac{\nu}{\ell}\right\}}.$$

To show $\frac{c}{2}(\frac{\overline{\gamma_{6,b}}-c}{3})\frac{\nu}{\ell} \in \mathbb{Z}$ it suffices to show $\frac{\overline{\gamma_{6,b}}-c}{3} \in \mathbb{Z}$. Recall $\begin{pmatrix} 3a & \gamma_{6,b} \\ c/2 & \gamma_{6,d} \end{pmatrix} \in SL_2(\mathbb{Z})$, so $3a \cdot \gamma_{6,d} - \gamma_{6,b} \cdot \frac{c}{2} = 1$, which implies $\gamma_{6,b} \cdot (\frac{c}{2}) \equiv -1 \pmod 3$. This gives that $\overline{\gamma_{6,b}} - c \in 3\mathbb{Z}$. For example, if $\gamma_{6,b} \equiv 1 \pmod 3$ then $\frac{c}{2} = 3r + 2$ which implies that $c \equiv 1 \pmod 3$. It then follows

$$P_{m,s} \otimes (\tfrac{\cdot}{\ell})(\tau)\Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha_s \cdot \Big\{ \frac{g}{\ell} \sum_{\nu=0}^{\ell-1} \Big(\frac{\nu}{\ell}\Big) \cdot e^{\frac{2\pi i}{24}(\ell^2-1)\left\{\frac{-\gamma_{3,b}\cdot c-4}{3}\right\}\frac{\nu'}{\ell}} \cdot q^{v_3} \Big\} + \dots \tag{6.28}$$
$$= \alpha_s \Big(\frac{v_3}{\ell}\Big) q^{v_3} + \dots .$$

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