A P-ADIC SUPERCONGRUENCE CONJECTURE OF VAN HAMME

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Abstract. In this paper we prove a p-adic supercongruence conjecture of van Hamme by placing it in the context of the Beukers-like supercongruences of Rodriguez-Villegas. This conjecture is a p-adic analog of a formula of Ramanujan.

1. Introduction

Recently, van Hamme [vH] made several conjectures concerning p-adic analogs of several formulas of Ramanujan. In this paper we prove one of these conjectures by making a connection between it and one of the Beukers-like supercongruences discovered by Rodriguez-Villegas [FRV].

We begin with numbers Apery used in his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$:

$$A(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$
$$B(n) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k},$$

where

$$(a)_k := a(a+1)\cdots(a+k-1), \text{ and } \binom{n}{k} := \frac{(-1)^k(-n)_k}{k!},$$

are the standard notations for the raising factorial and binomial coefficent respectively. For p an odd prime, Beukers made the following two conjectures concerning these numbers and the coefficients of two modular forms:

$$A\Big(\frac{p-1}{2}\Big) \equiv a(p) \pmod{p^2}, \tag{1.1}$$

$$B\Big(\frac{p-1}{2}\Big) \equiv b(p) \pmod{p^2}, \tag{1.2}$$

where

$$\sum_{k=0}^{\infty} a(n)q^n := q \cdot \Pi_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 \in S_4(\Gamma_0(8)), \text{ and}$$
$$\sum_{k=0}^{\infty} b(n)q^n := q \cdot \Pi_{n=1}^{\infty}(1-q^{4n})^6 \in S_3(\Gamma_0(16), (\tfrac{-4}{d})), \ q := e^{2\pi i z}.$$

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Both of these conjectures have multiple guises. Another form of (1.2) is found in [vH]:

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{-\frac{1}{2}}{k}^3 \equiv b(p) \pmod{p^2}. \tag{1.3}$$

Beukers proved these modulo p, [Be1], [Be2]. For (1.1), a partial proof was given by Ishikawa [I], and a complete proof was given by Ahlgren and Ono [AO]. For (1.3), proofs have been given by Ishikawa [I], van Hamme [vH], and Ahlgren [A]. (For techniques that yield a computer free version of [A], see Mortenson [M3].) Finite field analogs of classical hypergeometric series [G] play large roles in [AO], [A], and [M3].

In [FRV], Rodriguez-Villegas discovered numerically a number of Beukers-like supercongruences. This was motivated by his joint work with Candelas and de la Ossa [COV], where they studied Calabi-Yau manifolds over finite fields. For proofs of some of these congruences, see [M1], [M2], [M3], and [K], where again the theory of finite field analogs of classical hypergeometric series plays a large role. (The supercongruence of [K] is also found in the list of conjectures in [vH]). For example, we have the following:

Theorem 1.1. [M1], [M2] *Let p be an odd prime $p \geq 5$, and denote by $\phi_p(x)$ the Legendre symbol modulo p. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2} \equiv \phi_p(-1) \pmod{p^2}.$$

It should be noted that the proof in [M1] is software package dependent, whereas the proof in [M2] is not.

The motivation for this research comes from a recent paper by McCarthy and Osburn [McO], where they prove the following conjecture of van Hamme [[vH], page 226, (A.2)]:

Conjecture. [vH] (A.2) If p is an odd prime, then

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1)\binom{-\frac{1}{2}}{k}^5 \equiv p \cdot b(p) \pmod{p^3}.$$

Upon examining their proof, one is immediately struck by the strong shadow of one of Beukers' supercongruences, e.g. (1.3). In other words, one finds,

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1)\binom{-\frac{1}{2}}{k}^5 \equiv p \cdot \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{-\frac{1}{2}}{k}^3 \pmod{p^3}.$$

This leads one to speculate as to the potential role of other Beukers-like supercongruences, and brings us to the goal of this paper. With this idea, Theorem 1.1, and the arsenal of transformation formulas for generalized hypergeometric series found in Bailey's tract [B], we prove the next supercongruence conjecture on van Hamme's list [[vH], page 226, (B.2)]:

Conjecture. [vH] (B.2) Let p be an odd prime, then

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1)\binom{-\frac{1}{2}}{k}^3 \equiv p \cdot \phi_p(-1) \pmod{p^3}.$$

This is the p-adic analog of the following formula of Ramanujan [[vH], page 226, (B.1)]:

$$\sum_{k=0}^{\infty} (4k+1)\binom{-\frac{1}{2}}{k}^3 = \frac{2}{\pi}$$

The paper is organized as follows. In section 3 we recall necessary background information and in section 4 we prove a technical lemma. In section 5, we prove van Hamme's conjecture (B.2).

2. Acknowledgements

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3. Background information

We begin this section by recalling a theorem of Morley.

Theorem 3.1. [Mo] *Let p be an odd prime. Then*

$$2^{2p-2} \equiv \phi_p(-1)\binom{p-1}{\frac{p-1}{2}} \pmod{p^3}.$$

We recall the gamma function as defined by Weierstrass and found in [[WW], Ch. XII].

Definition 3.2. Let $z \in \mathbb{C}$, then

$$\frac{1}{\Gamma(z)} := ze^{\gamma z}\Pi_{n=1}^{\infty}\Big\{\Big(1+\frac{z}{n}\Big)e^{-\frac{z}{n}}\Big\},$$

where γ is Euler's constant.

From this definition it is apparent that $\Gamma(z)$ is analytic except at the points $z = 0, -1, -2, \dots,$ where it has simple poles. We also recall some gamma function properties, which are found in [[WW], CH. XII].

Proposition 3.3. *Let $z \in \mathbb{C}$. Then the following are true,*

1. $\Gamma(1) = 1$,
2. $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$,
3. $\Gamma(z+1) = z\Gamma(z)$,
4. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$,
5. $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \Gamma(\frac{1}{2})\Gamma(2z)$.

Lastly, we recall some generalized hypergeometric series transformations as found in Bailey's tract [B]. The first may be viewed as a specialization of Whipple's famous ${}_7F_6$ transformation [B, p.28]. The specialization is

$$\begin{aligned}&{}_6F_5\left(\begin{matrix}a, & 1+\frac{1}{2}a, & b, & c, & d, & e\\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\,\Big|\, -1\right)\\ &=\frac{\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(1+a)\Gamma(1+a-d-e)}\,{}_3F_2\left(\begin{matrix}1+a-b-c, & d, & e\\ & 1+a-b, & 1+a-c\end{matrix}\,\Big|\, 1\right),\end{aligned} \tag{3.1}$$

where

$$_{n+1}F_n\left(\begin{matrix}a_0, & a_1, & \dots, & a_n\\ & b_1, & \dots, & b_n\end{matrix}\,\Big|\, t\right)=\sum_{k=0}^{\infty}\frac{(a_0)_k(a_1)_k\cdots(a_n)_k t^k}{k!(b_1)_k\cdots(b_n)_k}.$$

is the classical hypergeometric series.

We recall a transformation formula for a terminating ${}_3F_2$ [B, p.22, eq. 1]:

$$\begin{aligned}&\frac{\Gamma(e+m)\Gamma(f+m)}{\Gamma(s)\Gamma(e)\Gamma(f)}\cdot{}_3F_2\left(\begin{matrix}a, & b, & -m\\ & e, & f\end{matrix}\,\Big|\, 1\right)\\ &=\frac{(-1)^m\cdot\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-a)\Gamma(s-m)\Gamma(1-b-m)}\cdot{}_3F_2\left(\begin{matrix}e-b, & f-b, & -m\\ & 1-b-m, & s-m\end{matrix}\,\Big|\, 1\right)\end{aligned} \tag{3.2}$$

where $s:=e+f-a-b+m$, and m is a positive integer. Here we are careful, because we will eventually look at the limit as a goes to one.

4. A technical lemma

Here we show a technical lemma, which appears in the proof of Conjecture (B.2).

Lemma 4.1. *Let $n=2t+1$ be a postive odd integer, and $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Then*

$$\frac{\Gamma(1+\omega\frac{n}{2})\Gamma(1+\omega^2\frac{n}{2})}{\Gamma(\frac{1}{2}-\omega\frac{n}{2})\Gamma(\frac{1}{2}-\omega^2\frac{n}{2})}=\frac{2^{2n-2}}{\Pi_{k=1}^{\frac{n-1}{2}}\left((2k-1)^2+3n^2\right)}.$$

Proof. We use Proposition 3.3 (4) to write

$$\begin{aligned}&\frac{\Gamma(1+\omega\frac{n}{2})\Gamma(1+\omega^2\frac{n}{2})}{\Gamma(\frac{1}{2}-\omega\frac{n}{2})\Gamma(\frac{1}{2}-\omega^2\frac{n}{2})}\\ &=\frac{\pi}{\sin(-\pi\omega\frac{n}{2})}\frac{\pi}{\sin(-\pi\omega^2\frac{n}{2})}\frac{1}{\Gamma(-\omega\frac{n}{2})\Gamma(\frac{1}{2}-\omega\frac{n}{2})}\frac{1}{\Gamma(-\omega^2\frac{n}{2})\Gamma(\frac{1}{2}-\omega^2\frac{n}{2})}.\end{aligned}$$

Writing sine in terms of the exponential function and simplifying, we obtain

$$\begin{aligned}&\frac{\pi}{\sin(-\pi\omega\frac{n}{2})}\frac{\pi}{\sin(-\pi\omega^2\frac{n}{2})}\frac{1}{\Gamma(-\omega\frac{n}{2})\Gamma(\frac{1}{2}-\omega\frac{n}{2})}\frac{1}{\Gamma(-\omega^2\frac{n}{2})\Gamma(\frac{1}{2}-\omega^2\frac{n}{2})}\\ &=\frac{4\pi^2}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}\cdot\frac{1}{\Gamma(-\omega\frac{n}{2})\Gamma(\frac{1}{2}-\omega\frac{n}{2})}\frac{1}{\Gamma(-\omega^2\frac{n}{2})\Gamma(\frac{1}{2}-\omega^2\frac{n}{2})}.\end{aligned}$$

Using the duplication formula of Proposition 3.3 (5), yields

$$\begin{aligned}\frac{4\pi^2}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}&\cdot\frac{1}{\Gamma(-\omega\frac{n}{2})\Gamma(\frac{1}{2}-\omega\frac{n}{2})}\frac{1}{\Gamma(-\omega^2\frac{n}{2})\Gamma(\frac{1}{2}-\omega^2\frac{n}{2})}\\ &=\frac{4\pi^2}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}\cdot\frac{2^{-\omega n-1}}{\Gamma(\frac{1}{2})\Gamma(-\omega n)}\frac{2^{-\omega^2 n-1}}{\Gamma(\frac{1}{2})\Gamma(-\omega^2 n)}\\ &=\frac{\pi}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}\cdot\frac{2^n}{\Gamma(-\omega n)\Gamma(-\omega^2 n)},\end{aligned}$$

where in the last step we have have used $\Gamma(\frac{1}{2})^2=\pi$. Recalling $n=2t+1$ where t is a nonnegative integer, we find

$$\begin{aligned}&\frac{\pi}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}\cdot\frac{2^n}{\Gamma(-\omega n)\Gamma(-\omega^2 n)}\\ &\quad=\frac{\pi}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}\cdot\frac{2^n}{\Gamma(t+\frac{1}{2}-\frac{n}{2}i\sqrt{3})\Gamma(t+\frac{1}{2}+\frac{n}{2}i\sqrt{3})}\\ &\quad=\frac{\pi}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}\cdot\frac{2^n}{\Pi_{k=1}^{\frac{n-1}{2}}((k-\frac{1}{2})^2+\frac{3}{4}n^2)\Gamma(\frac{1}{2}-\frac{n}{2}i\sqrt{3})\Gamma(\frac{1}{2}+\frac{n}{2}i\sqrt{3})}\\ &\quad=\frac{1}{e^{\pi\sqrt{3}\frac{n}{2}}+e^{-\pi\sqrt{3}\frac{n}{2}}}\cdot\frac{2^n\sin(\frac{\pi}{2}-\frac{\pi n}{2}i\sqrt{3})}{\Pi_{k=1}^{\frac{n-1}{2}}((k-\frac{1}{2})^2+\frac{3}{4}n^2)}=\frac{2^{n-1}}{\Pi_{k=1}^{\frac{n-1}{2}}((k-\frac{1}{2})^2+\frac{3}{4}n^2)},\end{aligned}$$

where the penultimate line follows from repeated application of Proposition 3.3 (3), and the ultimate line follows from Proposition 3.3 (4). The conclusion of the lemma easily follows. □

5. Proof of Conjecture (B.2)

Using an idea of McCarthy and Osburn [McO], we make the following choice of variables in equation (3.1). (An idea akin to this is also found in [vH]). We let $a=\frac{1}{2},\ b=\frac{1}{2}-\omega\frac{p}{2},\ c=\frac{1}{2}-\omega^2\frac{p}{2}, d=\frac{1}{2}-\frac{p}{2}, e=1$ where $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, to produce

$$\begin{aligned}&{}_6F_5\left(\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}-\omega\frac{p}{2}, & \frac{1}{2}-\omega^2\frac{p}{2} & \frac{1}{2}-\frac{p}{2}, & 1\\ & \frac{1}{4}, & 1+\omega\frac{p}{2}, & 1+\omega^2\frac{p}{2}, & 1+\frac{p}{2}, & \frac{1}{2}\end{matrix}\ \Big|\ -1\right)\\ &\quad=\frac{\Gamma(1+\frac{p}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})\Gamma(\frac{p}{2})}\,{}_3F_2\left(\begin{matrix}\frac{1}{2}-\frac{p}{2}, & \frac{1}{2}-\frac{p}{2}, & 1\\ & 1+\omega\frac{p}{2}, & 1+\omega^2\frac{p}{2}\end{matrix}\ \Big|\ 1\right).\end{aligned}$$

This one can consider $\pmod{p^3}$, and by using appropriate gamma function properties, one obtains

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\binom{-\frac{1}{2}}{k}^3\equiv p\cdot\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2}-\frac{p}{2})_k(\frac{1}{2}-\frac{p}{2})_k}{(1+\omega\frac{p}{2})_k(1+\omega^2\frac{p}{2})_k}\pmod{p^3}.\tag{5.1}$$

We now focus on the coefficient of p on the right hand side of equation (5.1). We recall equation (3.2) and let $a=1+\epsilon,\ b=\frac{1}{2}-\frac{p}{2}, m=\frac{p-1}{2},\ e=1+\omega\frac{p}{2}, f=1+\omega^2\frac{p}{2}$, to obtain

$$\frac{\Gamma(\frac{1}{2}-\omega^2\frac{p}{2})\Gamma(\frac{1}{2}-\omega\frac{p}{2})}{\Gamma(\frac{p}{2}-\epsilon)\Gamma(1+\omega\frac{p}{2})\Gamma(1+\omega^2\frac{p}{2})}\cdot{}_3F_2\left(\begin{matrix}1+\epsilon, & \frac{1}{2}-\frac{p}{2}, & \frac{1}{2}-\frac{p}{2} \\ & 1+\omega\frac{p}{2}, & 1+\omega^2\frac{p}{2}\end{matrix}\mid 1\right)$$
$$=\frac{(-1)^{\frac{p-1}{2}}\cdot\Gamma(-\epsilon)\Gamma(\frac{1}{2}+\frac{p}{2})}{\Gamma(-\epsilon)\Gamma(\frac{1}{2}-\epsilon)\Gamma(1)}\cdot{}_3F_2\left(\begin{matrix}\frac{1}{2}-\omega^2\frac{p}{2}, & \frac{1}{2}-\omega\frac{p}{2}, & \frac{1}{2}-\frac{p}{2} \\ & 1, & \frac{1}{2}-\epsilon\end{matrix}\mid 1\right).$$

Isolating the hypergeometric series on the left, taking the limit as ϵ goes to 0, and then examining everything $\pmod{p^3}$ yields,

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2}-\frac{p}{2})_k(\frac{1}{2}-\frac{p}{2})_k}{(1+\omega\frac{p}{2})_k(1+\omega^2\frac{p}{2})_k}$$
$$\equiv(-1)^{\frac{p-1}{2}}\cdot\frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{p}{2})\Gamma(1+\omega\frac{p}{2})\Gamma(1+\omega^2\frac{p}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\omega\frac{p}{2})\Gamma(\frac{1}{2}-\omega^2\frac{p}{2})}\cdot\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2}\pmod{p^3}.$$

However, we need only be concerned with the above equation $\pmod{p^2}$. Employing Proposition 3.3 (3) and (5) produces

$$\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2}-\frac{p}{2})_k(\frac{1}{2}-\frac{p}{2})_k}{(1+\omega\frac{p}{2})_k(1+\omega^2\frac{p}{2})_k}$$
$$\equiv(-1)^{\frac{p-1}{2}}\cdot 2^{1-p}(p-1)!\cdot\frac{\Gamma(1+\omega\frac{p}{2})\Gamma(1+\omega^2\frac{p}{2})}{\Gamma(\frac{1}{2}-\omega\frac{p}{2})\Gamma(\frac{1}{2}-\omega^2\frac{p}{2})}\cdot\sum_{k=0}^{\frac{p-1}{2}}\frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2}\pmod{p^2}.$$

From Lemma 4.1, we have

$$\begin{aligned}
&\sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2}-\frac{p}{2})_k(\frac{1}{2}-\frac{p}{2})_k}{(1+\omega\frac{p}{2})_k(1+\omega^2\frac{p}{2})_k}\\
&\quad\equiv (-1)^{\frac{p-1}{2}} \cdot 2^{1-p}(p-1)! \cdot \frac{2^{2p-2}}{\Pi_{k=1}^{\frac{p-1}{2}}((2k-1)^2+3p^2)} \cdot \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2} \pmod{p^2}\\
&\quad\equiv (-1)^{\frac{p-1}{2}} \cdot \frac{2^{p-1}(p-1)!}{\Pi_{k=1}^{\frac{p-1}{2}}((2k-1)^2)} \cdot \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2} \pmod{p^2}\\
&\quad\equiv (-1)^{\frac{p-1}{2}} \cdot 2^{2p-2} \cdot \binom{p-1}{\frac{p-1}{2}}^{-1} \cdot \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2} \pmod{p^2}\\
&\quad\equiv \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2} \pmod{p^2},
\end{aligned}$$

where the last line follows from Morley's result Theorem 3.1.

We now return to equation (5.1) and see

$$\begin{aligned}
\sum_{k=0}^{\frac{p-1}{2}} (4k+1)\binom{-\frac{1}{2}}{k}^3 &\equiv p \cdot \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2}-\frac{p}{2})_k(\frac{1}{2}-\frac{p}{2})_k}{(1+\omega\frac{p}{2})_k(1+\omega^2\frac{p}{2})_k} \pmod{p^3}\\
&\equiv p \cdot \sum_{k=0}^{\frac{p-1}{2}} \frac{(\frac{1}{2})_k(\frac{1}{2})_k}{k!^2} \pmod{p^3} \equiv p \cdot \phi_p(-1) \pmod{p^3}
\end{aligned}$$

where the last congruence follows from Theorem 1.1.

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