STUDENT'S SOLUTIONS MANUAL

Introduction to Linear Programming

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This manual includes:
corrections to the textbook,
additional references,
answers and solutions for exercises the textbook,
tips, hints, and remarks.

Contents

Chapter 6. Transportation Problems

Chapter 7. Matrix Games

Chapter 8. Linear Approximation

Corrections

This manual includes Answers to Selected Exercises (pages 305–317 of the first print of textbook) with some corrections and gives more solutions and answers. Note that the exercises may have many correct solutions and even several correct answers.

Here are some other corrections to the first print.

Dedication page (page iv). Replace my by our.

In **Contents,** pages v and vi, replace page numbers:
§6 52 → 54, §17 180 → 179,, §20 210 → 211, §21 220 → 221,

In **Preface**, page vii, the last row, replace Thus by Thus, . On page ix, update the URL for Mathematical Programming Glossary to
http://glossary.computing.society.informs.org/
and the URL for SIAM Activity Group on Optimization to
http://www.siam.org/activity/optimization/.

On page ix, replace
However, the students in class are allowed to use any hardware and software they like, even during tests.
by
However, the students are allowed to use any hardware and software they like, with proper references, in homework.

Page 2, line 11, replace the last comma by period.

Page 3, line 17, insert finitely many after of.

Page 4. On line 15 from below, add minus before 7. On line 3 from below., add a period after 1.3.

Page 5, line 14. Remove one comma from ",,".

Page 6, line 9 from below. Add period after 1.8.

Page 7, line 6. Replace 2 by 3.

Page 9. On line 1, delete "s" in "differents". On line 4 from below, replace rediscoved by rediscovered

Page 10. In Exercise 23, delete the period.

Page 11. There are two Exercises 57. Both solved below. Replace **36–42** by **36-43**. On line 5 from below, replace linear by a linear form.

Page 13, line 8 after the table. Remove space before the question mark.

Page 14. On line 5, insert a period after (mg). On the last line, insert dual after the and replace 4 by 5.

Page 16. line 5 from below. insert a period in the and. On the last line, insert a comma after Thus.

Page 17. Add period in the end of display. In the table, replace Aarea by Area and .6 by .8.

Page 19, the last line. Add period in the end .

Page 20. On line 7 and line 7 from below, insert a period in the end . On line 12 from below, insert a comma after Thus.

Page 21, line 14. Replace to worker by for worker.

Page 22, line 14. Insert it after Although.

Page 23. On lines 2 and 5, replace §1 of Chapter 2 by §3. On line 16, replace 2.6 by 2.5.

Page 26. On line 7, replace *nonzero* by *positive*.

Page 27. Add a period after **Figure 3.5** and in the end of line 4 from below.

Page 28. On the first line after Figure 3.7, replace §3 by §12. Remove the periods after the names of Figures 3.7 and 3.8.

Page 30, line 12 from below. Replace min by max.

Page 31. Remove the period after the name of Figure 3.14. On the third line after Figure 3.14, replace 4.25 by 4/2. On the last line, replace §3 by §12.

Page 33, last line. Replace $y + y$ by $y + z$.

Page 35, line 14. Remove space between $\wedge$ and) .

Page 38, line 18. Replace that by than.

Page 39, line 14. Insert : in the end.

Page 40, line 8 from below. Replace 30 by 31

Page 41, Ex. 31 Replace if and only if by means that

Page 42, line 8 from below. Remove the comma before (.

Page 43, line 2 from below. Replace uses by use.

Page 46, line 9 from below. Delete the second period after 5.8.

Page 47, line 3 Replace prededing by preceding

Page 49. In 2 displayed matrices, replace I by 1. In the proof of Proposition 5.14, replace I_n by 1_n six times. On line 2 from below, replace multiple by scalar multiple and drop "by a number".

Page 50. On line 3, replace B by A. Change the last row of the matrix P to [1 0 0]. In the last row of the matrix D, drop 4. In the last row of the matrix D^{-1}, replace 8 by 2.

Page 51. Replace the matrices E and E^{-1} by their transposes.

Page 52, Ex.12–14. Replace "in the sense of Definition 1.3" by "(see page 4)".

Page 53, Ex.44. Delete !!.

Page 54. On line 5, replace system by systems. On line 17 from bottom, replace correspond by corresponds. On line 10 from bottom, delete set of.

Page 56. On line 19, replace cz by $-cz$. On line 4 from bottom, replace a by an. On the last line, delete the first comma.

Page 57. line 2 from bottom Replace system by systems.

Page 58. On line 4, replace system by systems. On line 14 from bottom, replace time by times. On line 13 from bottom, replace a by an.

Page 59. On line 9 replace a by an.

Page 61. On line 12 (left of 2nd row of the third matrix), replace $-3/2\cdot$ with $1/2\cdot$. On line 9 from bottom, replace -71 by 7. On line 5 from bottom, replace - 71 by + 7.

Page 62. On lines 4 and 18, replace -71 by 7. On line 7, drop to. On line 11, switch exactly and means. On line 14 from bottom, replace $-7/8$ by $+7/8$.

Page 63. On line 12 from bottom, replace $c'(Cz+d)+c''z$ by $c'y+c''(Cy+d)$. On line 11 from bottom, replace $c'C+c" = 0$ and $c'd = 0$ by $c''C + c' = 0$ and $c''d = b_0$. On line 10 from bottom, replace cd by $c''d$.

Page 64, line -3. Replace x_3 by x_2.

Page 66, line 19. Replace **36–38** by **36–39**.

Page 68. On line 7, delete the second "of." On line 15, replace necessary by necessarily.

Page 70. On line 15 from bottom, replace 7,4 by 7.4. On line 13 from bottom, delete the after equivalent. On line 5 from bottom, add s after eliminate.

Page 71. On line 3, replace Thick by Trick. On line 16, add s after constraint. On line 18, insert by before adding. On line 5 from bottom, replace in by is.

Page 72. On line 1, replace rid off by rid the program of. On line 4, replace know any upper or lower bonds by do not know any bounds. On line 19 from bottom, replace preeding by preceding.

Page 73. On line 11, replace $x + 2y - 1$ by $3x + 4y - 1$. On lines 18 and 20., replace ≥ 1 by ≤ -1.

Page 74, lines 17. Replace 7,1 by 7.1.

Page 75, line 10. Replace method by methods.

Page 76, line 6. Delete space between b and :.

Page 77. On line 2, replace system by systems. In the end of line 5 from bottom, delete $, y \geq 0$.

Page 78,. On line 5., add the after one of. In (8.5), replace 7 - 2 by 6 - 3.

Page 80. In the first paragraph, replace β by γ. On line 15, replace -1/17 by 1. In the end of line 10 from bottom, replace 3, 5 by 4, 7.

Page 81, line 3. Insert of after column.

Page 82, line 5. Insert our before system.

Page 85, line 5 from bottom. Replace is by are.

Page 86, line 1. Delete the period after e.

Page 87. In Ex.7, remove "=."

Page 89. Insert a period after **9.1**.

Page 90,. On line 18., replace $,z$ by $,\ z$. On line 6 from the bottom, remove the semicolon after sign.

Page 91, the first tableau. Replace $= w$ by $= -w$.

Page 93, two lines above the last tableau. Delete "is."

Page 95, line 3. Delete space between -1 and :.

Page 96. On line 1, Replace Ax by $-Ax$. On lines 2,6, 7, 7, replace y by u. On line 11 from bottom, replace y by u, v. On line 9 from bottom, replace Trick 7.7 by Trick 7.8.

Page 97. Insert period after the first tableau. On line 9, replace $Ax - u = b$ by $Ax - u = -b$.

Page 98. Delete the first line after the first tableau.

Page 99, line 4 from bottom. Replace x_7 by $= x_7$.

Page 100, the last line. Replace $\geq$ by $\leq$.

Pages 102–132, headings on even pages. Delete " :" after Chapter 4.

Page 102, line 5 from bottom. Replace $=\!\!\rightarrow$ by $\rightarrow$.

Page 105. On line 4, replace isthe by is the. In the first tableau, change *min* at right margin by min.On line 15 from bottom, replace -2and by -2 and. On line 8 from bottom, replace $2x_2$ by $2x_3$. On line 6 from bottom, replace min= by min = .

Page 106, the last row in the 4th tableau. Replace 1 0 1 1 by 1 4 1 1.

Page 107,. On line 11, drop - after x_3 to and replace x_2 by x_3. On line 21, replace previosly by previously.

Page 110., Exercises. Put the periods after **1** and **2** in boldface.

Page 111. Put the period after **3** in boldface. On line 2 from bottom, delete the first "one."

Page 112. On the last row of (11.2) and on the last line of the page, drop the last x. On line 9 from bottom, replace "column" by "row."

Page 113, line 6. from bottom. Put the colon in boldface and remove space before it

Page 114. Insert a period after the tableau.

Page 115, line 18. Replace "not necessary follows" by "does not necessarily follow."

Page 116. In Problem 11.6, replace in 10.4 by in 10.10. In the solution, replace Phase 2 by Phase 1.

Page 118. In the first tableau. right margin, move the period down to the last row. On the line above **Remark**, replace $x_4 = 1$ by $x_1 = 1$. On the second line from bottom, insert a space before If.

Page 119. On line 9, replace z be by z by.

Page 120. In Exercise 10, remove has. In Ex. 11, 12, 13 put the periods in boldface.

Page 121. On line 12, replace less by fewer. Remove the colon after the name of Figure 12.3.

Page 122. Remove the period after the name of Figure 12.4. On line 11 from bottom, replace x_2 by y_1.

Page 123. On 13 from bottom, add an after As. On line 4 from bottom, replace constraint by constraints.

Page 124, line 1. Remove the period after Definition.

Page 125. On line 12 from the bottom, replace Ax^t by Ax^T. On line 5 from bottom, insert is before empty.

Page 127. On line 11, drop of. On line 14, replace coulb be now by now could be. On line 16, replace that is by stated as. Add a period after **12.19**.

Page 128. On line 21 from bottom, replace "a adjacent" by "an adjacent." On lines 15 and 7 from bottom, replace "$=u$" by "$+b=u$."

Pages 133, line 10, Replace $+c_m x_m$ by $+c_m x_m + d$.

Pages 134–164, headings on even pages. Delete ":" after Chapter 5.

Page 135. In Definition, replace associated to by associated with. On the last line, replace min by max.

Page 136, line 9 from bottom. Replace previosly to by previously.

Page 137, line 9 from bottom. Replace ^{t} by ^{T} twice.

Page 138.Replace dv in the second displayed line (line 14) by $d=v$. On line 16, replace $\leq$ by $\geq$. On line 5 from bottom, add an before equality.

Page 139. On line 6, add the before equality. Switch Case 2 and Case 3 on lines 11 and 9 from bottom.

Page 140, line 17 from bottom. Replace in by on.

Page 141. Replace -2 in the third line of the first matrix and the second line of the second matrix by 2.

Page 143. On line 11, insert comma before but. On last line in (14.2), replace respectivel,y by respectively. In the last row of the last tableau (the objective function C), replace 8 10 15 5 60 by 10 15 5 60 8.

Page 144,. On line 16 from bottom. replace -1 by -50. On line 9 from bottom. Insert "an" in between"get" and "improvement"; insert "to" between "equal" and "280/297."

Page 145. On lines 6 and 10, replace 0.240 by 0.2458. In (14.5), insert - before ε_2. On the first line after (14.5), replace tableu by tableau. On the first and second lines after (14.5), replace "It is easy to compute now for which values of ε_i the tableau stays optimal" by "It is now easy to compute the values of ε_i for which the tableau stays optimal." On the ninth line after (14.5), replace stay by stays. On the tenth line after (14.5), replace 48.38 by 49.38.

Page 146, line 10 and 11 from bottom. Insert the after to.

Page 150, line 14 from bottom. Replace e by ε.

Page 151, line 12 from bottom. Replace row by raw.

Page 152, Theorem 14.15. Replace the lines 6–7 by:
value. Then P is a convex set, and, when parameters are in c (resp., in b), $f(t)$ is the minimum (resp., maximum) of a finite set of affine functions on P. So $f(t)$ is a piecewise affine and concave (resp., convex)

Page 153, line 10 from bottom. Replace $y >\geq$ by $y \geq$.

Page 154. On line 3 after (17.7), replace chose by choose. On line u in (14.20), replace -1/3 by 1/3.

Page 155, line 10. Replace i, and j by i, j, and k.

Page 156, line 4 from bottom. Replace low by lower.

Page 157, on right from the first tableau. Add space between all and $x_i \geq 0$.

Page 158, line 12 from bottom. Add the before duality.

Page 159. On line 6, replace Is by Does. On line 7, replace follows by follow. Add a period after **Remark 15.2**. Two lines later, replace $yA \geq c$ by $yA \leq c$.

Page 160. On line 14 from bottom, replace van you to by can. On line 10 from bottom, delete to. On line 5 from bottom, replace 2.1 by 2.2.

Page 161. On the right of tableau, replace $u \geq= 0$ by $u', u'' \geq 0$. On line 5 from bottom, add a comma after Bob.

Page 162. On line 4, replace b_o by b_0. On line 10, delete of after dropping.

Page 163, last line of the tableaux. Replace 105 by 122.

Page 164. Remove the period after **Remark**.

Page 165. In Exercise 8, replace $+-$ by $-$. On line 4, delete the period before *Hint*. Reduce the height of brackets in Exercises 9 and 10.

Page 166, line 6 from bottom. Replace transpcrtation by transportation.

Page 168, line 3 above the last table. Replace 2-by-3 table by 2-by-2 table.

Page 169, line 1 above the last table. Add a colon in the end.

Page 170. On line 2, insert a period in the end of the displayed formula. On line 3, replace 15.2 by 15.7. On line 9, replace $= b_j$ by $\geq b_j$ and switch m and n. On line 10, replace $= a_i$ by $\leq a_i$ and switch m and n. On lines 11 and 13, switch m and n.

Page 171. On line 17 from bottom, delete the comma before ").” On line 6 from bottom, insert that after so.

Page 172, line 4 after the 1st table. Replace row by column.

Page 173, the last matrix. Replace it by $\begin{bmatrix} 77 & 39 & 105 \\ 150 & 186 & 122 \end{bmatrix}$.

Page 174, **Figure 16.5**. Replace cost 1, 2, 2, 2 on the arrows by 77, 39, 186, 122.

Page 175. In **Table 16.6**, replace 15.2 by 15.7. In **Figure 16.7**, replace the cost 1, 2, 2, 2 on the arrows by 77, 39, 186, 122 and the potentials 0, 0, 1, 2, 2 at the nodes by 0, -147, 77, 39, -25.

Page 176, line 1 after **Table 16.9**. Replace previosly by previously.

Page 180. On line 4, replace 30) = by 50) =. On line 3 above Table 17.4, replace the second and by but. In the first row of Table 17.4, center (65), (60), and (50).

Page 181, the figure title. Add a period after **17.5**.

Page 183, the line above Figure 17.10. Add space between Figure and 17.10.

Page 184, line 4 after **Table 17.11**. Replace e by ε.

Page 188, the second figure title. Add a period after **17.21**.

Page 189. On top of Figure 17.23, replace $c = 25$ by $c = 35$. On the last line, replace ficticious by fictitious.

Page 191. On line 15, delete the period after **Exercises.**

Page 192. On line 7, delete then twice. On line 16 from bottom, switch “)” and “.”.

Page 193. On line 8, insert space between . and The. On line 9, replace problem we may by problem may. On line 3 in **Solution**, remove . between that is and , the. On line 4 in **Solution**, switch “)” and “.”.

Page 194, first table. Insert * as the last entry in the first row. Move * in the last row from the second position to the first position.

Page 195. In the third table, delete the last two asterisks. In the next line, replace “column does produce four” by “row does produce two”. On the last line, replace + 2 by +3.

Page 196. In the second row in first two tables, replace the first 3 entries (2) (3) (1) by (1) (2) (0). On line 4 after the first table, replace 6 by 4. On the line above the last table, drop of after along. In the last table, replace the potentials 6, 2, 1 at the left margin by 5, 3, 2. In the second row, replace the first 3 entries (3) (3) (2) by (2) (2) (1).

Page 197. In the first table, replace 1 (-1) $0-\varepsilon$ in the first row by $1-\varepsilon$ (-1) 0 and 3 | (0) (0) (-7) in the second row by 2 | $(-1)\varepsilon$ (-1) (-2). In the next line, replace (2, 3). Again $+\varepsilon = 0$. by (2, 1), and $\varepsilon = 1$. In the next line, replace "(1, 3) (no other choice this time)." by (2, 4). In the second table, replace 1 (0) (1) 0 in the first line by 0 (0) (1) 1 and (0) (1) 0 1 in the second line by 1 (1) 0 (1). In the last table replace the first two lines $\begin{array}{cccc} * & & & \\ & & & * \end{array}$ by $\begin{array}{cccc} & & & * \\ * & & & \end{array}$. Replace the next three lines by: The optimal value is min = 10. On line 4 from bottom, replace $[n$ by $(n$.

Page 200. On line 13, replace hin by him. On line 7 from bottom, replace paoff by payoff and delete the last period. On the second line from bottom, insert of before as.

Page 201. On line 2, replace the period by a colon. On line 8 from bottom, replace an by a. On the next line remove space between (and If.

Page 202, line 2 from bottom. Drop is after is.

Page 203. Remove the period after **Definition**.

Page 204, the first two displayed formulas. Place $p \in P$ under max and $q \in Q$ under min.

Page 205, line (2,3) in two matrices. Replace 1 1 -1 by 1 0 0. This makes both matrices skew symmetric.

Page 208. On line 3, replace previosly by previously.

Page 210, In Exercise 10., the empty entry means 0. On line 10 from bottom, replace (0. 3) by (0, 3). On line 9 from bottom, replace (1,2) by (1, 2).

Page 211, On line 20, replace) by]. On line 8 from bottom, replace Player 1 by She. On line 4 from bottom, replace win 1/4 from by lose 1/4 to. On line 3 from bottom, add ^{T} after] and replace Scissors by Rock.

Page 212. On line 14, add e to th. On line 11 from bottom, insert space before by in -1/2by.

Page 213, line 5 from bottom. Replace nodes by node.

Page 215, line 2 from bottom. Replace $,\mu'$ by $,\ \mu'$.

Page 216, line 19. Insert is between problem and solved.

Page 217, line 5 after the first matrix. Replace lows by lows us.

Page 218. On line 7, add the before game. On the last line, replace solutions by strategies.

Page 219. On line 12., replace A by M. On line 6 from bottom, replace).] by)].

Page 221. On line, remove the before blackjack. On line 12 from bottom, remove space before the question mark. On line 6 from bottom, replace loose by lose.

Page 223. On line 2, replace c3+ by c3 +. On line 2 from the last matrix, replace with r3 by with c3.

Page 224, line 2. Replace c2 by c3.

Page 226. On line 3, drop of. On line 17, replace c4,c5 by c4. c5.On line 4 from bottom, replace the second 1/4 by 1/2. On line 2 from bottom, replace 0.1 by 0, 1.

Page 227. On line 2, replace: [1,1,1] by [1,2,1] ; [0 by [1 . On line 5, replace 0.1 by 0, 1. In Exercise 5, replace games by game.

Page 229. On line 6, switch we and can. On line 8, insert is between it and not.

Pages 230–256, headings on even pages. Delete "." after Chapter 8.

Page 230. On line 5, replace k by m. On line 17, replace number by numbers.

Page 231. On line 9 and line 5 (twice) from bottom, replace k by m. On line 4 from bottom, replace tells that by tells us that.

Page 232. On line 9, replace to with by to do with. On line 13, replace suma by sumo. On line 16, replace "Survival" TV show by TV show "Survivor". On line 3 from bottom, replace "$5h+75$ for women and $w = 6h+76$" by "$5h-200$ for women and $w = 6h-254$". On the next line, also replace 75 by -200 and 76 by -254.

Page 233,. On line 13 from bottom, switch we and can. On line 7 from bottom, replace Three by The three.

Page 234. On line 6, insert h after d twice. On line 7, replace function by functions. On line 6 from bottom, replace NHI by NIH .

Page 235, last line. Replace x_1 by c_1.

Page 236. On line 10, insert the before l^1-approach. On line 8 from bottom, replace 18 by 19.

Page 238. On line 5, insert of in front of one. On line 8, replace kind by kinds. On line 10 from bottom, insert a between solve and system.

Page 239, line 1. Replace on by of.

Page 240. In **Remark**, insert the before literature twice. In Exercise 9, replace p by p twice. In Exercise 9, drop the last sentence (which repeats the previous one). There are two Exercises 12. Both solved below. In the last exercise, replace **13** by **14**.

Page 241. On the line above **Example 23.1**, replace Otherwise by When the columns of A are linearly independent. On line 4 from bottom, replace $A =$ by $A^T =$. On line 3 from bottom, replace $w =$ by $w^T =$. On the last line, switch a and b.

Page 242. On line line 5, replace $A =$ by $A^T =$. On line 6, insert ^{T} after w. On line 8, replace $\begin{bmatrix} b \\ a \end{bmatrix}$ by $\begin{bmatrix} a \\ c \end{bmatrix}$. On lines 16 and 12 from bottom, replace a by X.

Page 243,. On line 10, replace $a =$ by $b =$. On line 15, replace consider by considered. On line 12 from bottom, replace $e_i|$ by $|e_i|$. On lines 17 from bottom, replace know by known. On lines 15 from bottom, replace **best** by **Best**. On lines 13, 10, and 8 from bottom, replace a and a_j by X. On line 8 from bottom, replace t by u. On line 6 from bottom, add a period after 23.5.

Page 244. On lines 6 and 8 from bottom, delete vskip-5pt three times.

Page 246. On line 5, replace better by the best. On line 8, replace the last B by C.

Page 247. On line 11, add , before etc. On line 18, replace allowe by allow. On line 17, replace semicolumns by semicolons On lines 11 (7) and 2 from bottom, replace A^t by A^T.

Page 248, On line 4, drop (and replace) by }. On line 11, delete the space after Maple. On line 9 from bottom, replace of by at.

Page 249. On line 6, replace $p = 3$ by $p = 2$. On line 9, replace not so by not as. On line 14, replace in trash by in the trash.

Page 250. In Ex. 9, replace best$l$ best l. In Exercise 13, replace $+1/\alpha^{t+1}$ by $-(-1/\alpha)^{t+1}$. In Exercise 14, replace the first four periods by commas.

Page 251. On line 4, remove space before the comma.On line 16, replace questions by question. On line 17, insert of before \$5K. On line 22, replace those by these.

The link on line 3 from bottom does not work anymore. In

http://nbakki.hatenablog.com/entry/2014/02/15/000000

per capita chocolate consumption in Japan in 1982–2012 is given (in kg). But the numbers for 1985-1995 given there and in Example 24.1 are not quite the same. So history changes not only in novels and movies.

In

http://www.chocolate-cocoa.com/english/pdf/index_002.pdf

the number for 2013 is given, 1.93 kg.

Page 252. On line 9 from bottom., put the period after **24.3** in boldface. On line 8 from bottom., replace billions by billion. Remove the period after the name of Figure 24.3.

Page 253. On line 16, replace **Example 24.3** by **Example 24.4**. On line 14 from bottom, drop one from. On line 7 from bottom, replace this year by of the year. On line 5 from bottom, replace intitial by initial. On line 4 from bottom, replace sufficiantly by sufficiently.

Page 254, line 8. Replace accept by except.

Page 255. On line 6, replace liner by linear. On line 11, replace date by data. The link on line 12 does not work anymore. The production numbers for 1990-1993 in Exercise 1 were changed after publication of the textbook. Delete the minus in -1984. On line 19, italicize x and y.

Page 256. In Ex. 3, replace 24.3 by 24.4. On line 4 from bottom., replace l_p by l^p.

Page 258. On line 15 from bottom, drop "a". On line 14 from bottom, replace "then" by "than".

Page 259. On line 14, drop a from a a. On line 16, drop of.

Page 260. On line 8 , switch . and). On line 10 , switch . and]. On line 4 from bottom, delete the first the.

Page 262. On line 10, remove the last).On line 10 from bottom, replace = 0.] by = 0]. On line 5 from bottom, replace otherwize by otherwise.

Page 263. On line 6 and 3 from bottom, replace Lipshitz by Lipschitz.

Page 264. On lines 17, replace Lipshitz by Lipschitz and add is after or. On line 14 from bottom, replace [V2] by [V]. On line 2 from bottom, insert the second) before /.

Page 265. On line 15, replace $(0 - g(x_1)$ by $(0 - g(x_1))$. On line 7 from bottom, replace Lipshitz by Lipschitz.

Page 267. On the last line, insert the second) before +.

Page 268. On line 8 from bottom, drop) . On line 5 from bottom, insert) after x_{t+1}.

Page 269. Replace the heading by that from page 267.

Page 271. Add period in the end of the second display.

Page 273. In head, replace A3. ... by A4. ... On line 11 from bottom, delete to.

Page 275. On line 7, insert) before the comma. In (A4.6), replace the second $F(w)$ by w.

Page 276. On line 8, delete the comma in the end. On line 9 from bottom, replace get better by get a better.

Page 277. On line 16 from bottom, delete the comma after methods. On line 14 from bottom, insert the second] after L.

Page 282. On line 11 from bottom, delete the third).

Page 283, line 11. Replace , The by . The . On line 18 from bottom, insert) between | and / .

Page 284, line 16. Delete the last).

Page 285. In head, replace A6. ... by A5. ... . On line 4, replece f by h. On lines 8 and 9, drop) from)). On line 9, drop −. On line 10, insert the before setting.

Page 286, line 1 and 2. Replace Pertubation by Perturbation .

Page 287. In head, replace A6. ... by A7.Goal Programming. On line 12 from bottom, add) in the end of line.

Page 289, line 5 from bottom. Replace , While by . While .

Page 290, line 6 from bottom. Replace point ,and by point, and.

Page 291, line 10. Replace (ee by (see .

Page 292, line 11 from bottom. Insert $\leq$ between x_i and e_i.

Page 293. On line 8, replace $y + 2$ bt y_2. On line 5 from bottom, replace by by be .

Page 295, Theorem A10.2. Replace s by n. Replace "$n\rceil = m$" by "$i\rceil = n$."

Page 296. On line 6, replace ** by *. On line 18 from bottom, replace previosly by previously. On line 10 from bottom, replace . then by , then. On line 5 from bottom, replace $o(11)$ by $o(1)$.

Page 297. In head, replace A11. … by A10. … . On line 3, replace tupple by tuple. On line 2 from bottom, replace $F_)$ by F_0.

Page 299. On line 12, replace Transportation by The transportation. On line 15 from bottom, remove space before the semicolon.

Page 301. Remove space before comma in [B1]. Remove space after period in [DL]. Insert comma after C. in [C1]. Add space before "and" in [DL].

Page 302. In [FMP], insert space before C. In [FSS], replace Forg by Forgó and Szp by Szép.

Page 303. Remove space after period in [K4]. Remove space before comma in [K5]. In [L], insert space before G. Remove space after period in [NC].

Page 304. In [S3], insert space before M. In [VCS], insert space before I. In [V], replace V. by N.

Page 318 (index). Replace assignment problem,, by assignment problem,. Delete space after Dantzig.

Page 319. Replace "48 ," by "48,", "." by ",", inconsistant by inconsistent, "Karush-Kuhn-Tucker, 280" by "Karush-Kuhn-Tucker, 260", "KKT conditions, 280" by "KKT conditions, 260", klein by Klein, Lagrange multiplies by Lagrange multipliers.

Page 320. Switch "operations research" and "operational research" and replace "." by ",". Add "," after "payoff matrix". Add space before = on lines 3 and 4 in the right column. Add space after = on lines 3 in the right column. Replace the period by a comma on the last row in the right column.

Page 321. Replace the period by a comma on the first row in the left column. Replace "." by "," 1144 by 144, and TCP by TSP.

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Chapter 1. Introduction

§1. What Is Linear Programming?

T1. *What is a number?*

Unless said otherwise, a *number* means a *real number*. Here are examples of numbers: 0. 1, 2, -1, -2/3, 0.5, $\sqrt{2}, \pi$. e. The first 4 numbers are integers. The first 6 numbers are rational numbers. The last 3 numbers are irrational.

The equation $x^2 = -1$ has no solutions because $x^2 \geq 0$ for every (real) number x.

A bad news is that the concept of a real number is complicated. A good news is that Linear Programming can be done in rational numbers. That is, if all given numbers are rational numbers, then the answer can be given in rational numbers. This is related to the fact that any system of linear equations with rational coefficients can be solved in rational numbers.

T2 *Division by* 0

The number 0 is special, see **0 (number)** and **Division by zero** in Wikipedia. In particular, division by 0 is not allowed in our class. E.g., take the correct statement $0 \cdot 0 = 0 \cdot 1$. Dividing by 0, we obtain the wrong statement $0 = 1$. By the way, both $0 \cdot x = 0$ and $0 \cdot x = 1$ are linear equations for x in standard form. But every x is a solution for the first equation while the second equation has no solutions.

T3. *Inequalities*

In linear inequalities, besides the signs "$\leq$" and "$\geq$", the signs "$>$" and "$<$" are allowed (sometimes, "$\neq$" is allowed too). However the signs "$<$," "$>$", and "$\neq$" are not allowed in linear programs. So the statement "More formally, linear programming is a technique for the optimization of a linear objective function, subject to linear equality and linear inequality constraints." in

http://en.wikipedia.org/wiki/Linear_programming

is wrong

T4. *Polynomials*

For those who know polynomials, an affine function is a function given by a linear polynomial. A linear form is a homogeneous linear polynomial.

1. True.
3. True.
5. True. This is because for real numbers any square and any absolute value are nonnegative.
7. False. For $x = -1,\ 3(-1)^3 < 2(-1)^2$.
8. False (see Definition 1.4).
9. False (see Example 1.9 or 1.10).
11. False. For example, the linear program

$$\text{Minimize } x + y \text{ subject to } x + y = 1$$

has infinitely many optimal solutions.

13. True. It is a linear equation. A standard form is $4x = 8$ or $x = 2$.
15. No. This is not a linear form, but an affine function.
16. Yes, if z is independent of x, y.
17. Yes if a and z do not depend on x, y.
18. No (see Definition 1.1).
19. No. But it is equivalent to a system of two linear constraints.
21. Yes. We can write $0 = 0 \cdot x$, which is a linear form.
22. True if y is independent of x and hence can be considered as a given number; see Definition 1.3.
23. Yes if a, b are given numbers. In fact, this is a linear equation.
25. No. We will see later that any system of linear constraints gives a convex set. But we can rewrite the constraint as follows $x \ge 1$ OR $x \le -1$. Notice the difference between OR and AND.
27. See Problem 6.7.
28. We multiply the first equation by 5 and subtract the result from the second equation:

$$\begin{cases} x + 2y = 3 \\ \quad\ \ -y = -11. \end{cases}$$

Multiplying the second equation by -1, we solve it for y. Substituting this into the first equation, we find x. The answer is

$$\begin{cases} x = -19 \\ y = \ \ 11. \end{cases}$$

29. $x = 3 - 2y$ with an arbitrary y.
31. min $= 0$ at $x = y = 0, z = -1$. All optimal solutions are given as follows: $x = -y, y$ is arbitrary, $z = -1$.

33. max $= 1$ at $x = 0$.

35. min $= 0$ at $x = -y = 1/2, z = -1$.

37. No. This is a linear equation.

38. No. Suppose $x + y^2 = ax + by$ with a, b independent of x, y. Setting $x = 0, y = 1$ we find that $b = 1$. Setting $x = 0, y = -1$ we find that $b = -1$.

39. No.

41. Yes.

43. No.

44. No. Suppose that xy is an affine function $ax + by + c$ of x, y. Setting $x = y = 0$, we find that $c = 0$. Setting $x = 0, y = 1$, we find that $b = 0$. Setting $x = 1, y = 0$, we find that $a = 0$. Setting $x = y = 1$, we find that $1 = 0$.

45. Yes.

47. Yes.

49. No.

50. No, this is a linear form.

51. Yes.

53. Yes. In fact, this is a linear equation.

55. No. This is not even equivalent to any linear constraint with rational coefficients.

57. No, see Exercise 44.

57 (58 in the next print). Let $f(x, y) = cx + dy$ be a linear form. Then $f(ax, ay) = cax + day = a(cx + dy) = af(x, y)$ for all a, x, y and $f(x_1 + x_2, y_1 + y_2) = c(x_1 + x_2) + d(y_1 + y_2) = cx_1 + dy_1 + cx_2 + dy_2 = f(x_1, y_1) + f(x_2, y_2)$ for all x_1, x_2, y_1, y_2.

58 (59 in the next print). By additivity and proportionality, $f(x, y) = f(x, 0) + f(0, y) = f(1, 0)x + f(0, 1)y$.

59 (60 in the next print). min $= 2^{-100} - 1$ at $x = 0, y = 0, z = 3\pi/2, u = -100, v = -100$. In every optimal solution, x, y, u, v are as before and $z = 3\pi/2 + 2n\pi$ with any integer n such that $-16 \leq n \leq 15$. So there are exactly 32 optimal solutions.

§2. Examples of Linear Programs

T1. *On Example* 2.1

See Dietary Reference Intake in Wikipedia about the recommended dietary allowances.

T2. *On Example* 2.2

Since 1866, the nickel, the five-cent US coin, is made from cupronickel (also known as copper-nickel) composed of 75% copper and 25% nickel,.

2. min $= 1.525$ at $a = 0, b = 0.75, c = 0, d = 0.25$

4. Let x be the number of quarters and y the number of dimes we pay. The program is

$$25x + 10y \to \text{min, subject to}$$
$$0 \le x \le 100,\ 0 \le y \le 90, 25x + 10y \ge C \text{ (in cents)}, x, y \text{ integers.}$$

This program is not linear because the conditions that x, y are integers. For $C = 15$, an optimal solution is $x = 0, y = 2$. For $C = 102$, an optimal solution is $x = 3, y = 3$ or $x = 1, y = 8$. For $C = 10000$, the optimization problem is infeasible.

5. Let x, y be the sides of the rectangle. Then the program is

$$xy \to \text{max, sbject to}$$
$$x \ge 0,\ y \ge 0, 2x + 2y = 100.$$

Since $xy = x(50 - x) = 625 - (x - 25)^2 \le 625$, max $= 625$ at $x = y = 25$.

7. We can compute the objective function at all 24 feasible solutions and find the following two optimal matchings: Ac, Ba, Cb, Dd and Ac, Bb, Ca, Dd with optimal value 7.

8. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9 + 9 + 7 + 9 + 9 = 43$ for the objective function. This bound cannot be achieved because of a conflict over c (the third column). So max ≤ 42. On the other hand, the matching Aa, Bb, Cc, De, Ed achieved 42, so this is an optimal matching.

9. Choosing a maximal number in each row and adding these numbers, we obtain an upper bound $9+9+9+9+8+9+6 = 59$ for the objective function. However looking at B and C, we see that they cannot get $9 + 9 = 18$ because of the conflict over g. They cannot get more than $7 + 9 = 16$. Hence, we have the upper bound max ≤ 57. On the other hand, we achieve this bound 57 in the matching Ac, Bf, Cg, De, Eb, Fd, Ga.

11. Let c_i be given numbers. Let c_j be an unknown maximal number (with unknown j). The linear program is

$c_1x_1 + \cdots + c_nx_n \to \max$, all $x_i \geq 0$, $x_1 + \cdots + x_n = 1$.

Answer: $\max = c_j$ at $x_j = 1, x_i = 0$ for $i \neq j$.

T3. *Optimization and Science*

In science, any behavior is often explained as optimization. Here are two quotes from Action (physics) in Wikipedia:

"Classical mechanics postulates that the path actually followed by a physical system is that for which the action is minimized, or, more generally, is stationary." "The action is defined by an integral, and the classical equations of motion of a system can be derived by minimizing the value of that integral."

Here is a quote from Rational choice theory in Wikipedia::

"The rational agent is assumed to take account of available information, probabilities of events, and potential costs and benefits in determining preferences, and to act consistently in choosing the self-determined best choice of action."

If we call somebody "irrational," this usually means that we do not understand what his/her objective function is.

T4. *Optimization and Engineering*

Here is a quote from Engineering in Wikipedia: "If multiple options exist, engineers weigh different design choices on their merits and choose the solution that best matches the requirements. "

T5. *Advanced remarks about Diet Problem.* Sometimes RDAs are related to the total calories intake. If they are affine functions of the intake, we still have a linear program.

Sometimes buying food involves additional costs like cost of gas (when we go to supermarket by car) or shipping cost (when we buy food online) which may depend on the total weight of food. When the additional cost is an affine fucyion of the weight, it can be included to the total cost and we still have a linear program.

Suppose we consider buying food in one of 3 supermarkets. So we have 3 linear grograms to solvr (then we choose the best optimal value). By an advanced trick, we can write 3 linear programs as one linear program.

§3. Graphical Method

T1. *SSN*

Presently you are supposed to keep your SSN secret. So use your PSU ID instead.

1. Let SSN be 123456789. Then the program is
$-x \to \max, 7x \le 5, 13x \ge -8, 11x \le 10$.
Answer: $\max = 8/13$ at $x = -8/13$.

2. Let SSN be 123456789. Then the program is $f = x - 3y \to \min, |6x + 4y| \le 14, |5x + 7y| \le 8, |x + y| \le 17$.
Answer: $\min = -22$ at $x = -65/11, y = 59/11$.

3. Let SSN be 123456789. Then the program is
$x + 2y \to \min, |12x + 4y| \le 10, |5x + 15y| \le 10, |x + y| \le 24$.
Answer: $\min = -25/16$ at $x = -11/16, y = -7/16$.

4. The first constraint is equivalent to 2 linear constraints $-7 \le x \le 3$. The feasible region for the second constraint is also an interval, $-8 \le x \le 2$. The feasible region for the linear program is the interval $-7 \le x \le 2$. In Case (i), the objective function is an increasing function of x and reaches its maximum 14 at the right endpoint $x = 2$. In Case (ii), the objective function is a decreasing function of x and reaches its maximum 63 at the left endpoint $x = -7$. In Case (ii),

$$\max = \begin{cases} 2b \text{ at } x = 2 & \text{if } b > 0, \\ 0 \text{ when } -7 \le x \le 2 & \text{if } b = 0, \\ -7b \text{ at } x = -7 & \text{if } b < 0. \end{cases}$$

5. $\min = -72$ at $x = 0, y = -9$

6. The objective function is not defined when $y = 0$. When $y = -1$ and $x \to \infty$, we have $x/y \to -\infty$. So this minimization problem is unbounded, $\min = -\infty$.

7. $\min = -1/4$ at $x = 1/2, y = -1/2$ or $x = -1/2, y = 1/2$.

9. $\max = 1$ at $x = y = 0$

11. $\max = 22$ at $x = 4, y = 2$

13. The program is unbounded.

14. The feasible region van be given by 4 linear constraints: $-5 \le x \le 0, 2 \le y \le 3$. It is a rechtangle with 4 corners $[x, y] = [0, 3], [-5, 3][-5, -2], [0, -2]$. The objective function is not affine. Its level $|x| + y^2 = c$ is empty when $c < 0$, is a point when $c = 0$, and is made of 2 parabola pieces when $c > 0$. It is clear that $\max = 14$ at $x = -5, y = 3$. The optimal solution is unique.

15. max $= 3$ at $x = y = 0, z = 1$. See the answer to Exercise 11 of §2.

Chapter 2. Background

§4. Logic

T1. *Every or All*

The words "every" and "all" in English are not always interchangeable. Here is an example from

http://www.perfectyourenglish.com/usage/all-and-every.htm :

All children need love.

but

Every child needs love.

See

http://www.bbc.co.uk/worldservice/learningenglish

/grammar/learnit/learnitv266.shtml

for more information.

T2. *Inclusive or Exclusive*?

In modern English "or" is usually inclusive. A possible counter example is "Every entry includes a soup or salad." For exclusive "or" (xor), we usually use "either ... or" construction. Compare the following two statements about men in a town with one barber:

"Every man shaves himself or is shaved by a barber"

and

"Every man either shaves himself or is shaved by the barber" (Barber Paradox).

T3. $0 = 1$ *implies everything*?

Bertrand Russell, in a lecture on logic, mentioned that in the sense of material implication, a false proposition implies any proposition.

A student raised his hand and said "In that case, given that 1 = 0, prove that you are the Pope."

Russell immediately replied, "Add 1 to both sides of the equation: then we have $2 = 1$. The set containing just me and the Pope has 2 members. But $2 = 1$, so it has only 1 member; therefore, I am the Pope."

T4. $0 = 1$?

Some of you could be familiar with the integers modulo n. In the case $n = 1$, we have $0 = 1$.

T5. *Om Example* 4.3

The list in Example 4.3 is far for being complete. Here are more items involving subordinating conjunctions:

$x \geq 0$ as long as $x \geq 2$,
$x \geq 0$ now that $x \geq 2$,
$x \geq 0$ whenever $x \geq 2$,
$x \geq 0$ once $x \geq 2$,
$x \geq 0$ while $x \geq 2$,
$x \geq 0$ as $x \geq 2$,
$x \geq 0$ rather than $x \geq 2$,
$x \geq 0$ after $x \geq 2$.

1. False. For $x = -1$, $|-1| = 1$.
3. False. For $x = -10$, $|-10| > 1$.
5. True. $1 \geq 0$.
7. True. $2 \geq 0$.
9. True. The same as Exercise 7.
11. False. $1 \geq 1$.
13. True. $5 \geq 0$.
15. False. For example, $x = 2$.
17. True. Obvious.
19. False. For example, $x = 1$.
21. True. $1 \geq 0$.
22. Yes, we can.
23. Yes. $10 \geq 0$.
25. No, it does not. $(-5)^2 > 10$.
27. True.
29. False. The first condition is stronger than the second one.
30. False. The converse is true.
31. True.
33. (i) $\Rightarrow$ (ii),(iii), (iv).
35. (i) $\Rightarrow$ (iii).
37. (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii)
39. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii).
40. given that, assuming that, supposing that, in the case when, granted that.
41. "only if"
42. This depends on the definition of *linearfunction*.
43. No. $x \geq 1, x \leq 0$ are two feasible constraints, but the system is infeasible.
44. False.

45. False. Under our conditions, $|x| > |y|$.

47. Correct (add the two constraints in the system and the constraint $0 \leq 1$).

49. No, it does not follow.

51. Yes, it does. Multiply the first equation by -2 and add to the second equation to obtain the third equation.

§5. Matrices

1. $[2, 1, -6, 6]$

3. -14

5. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 3 \\ -2 & -4 & 0 & 6 \\ 4 & 8 & 0 & -12 \end{bmatrix}$

7. $(-14)^2 \cdot A^T B = \begin{bmatrix} 0 & -196 & -392 & 784 \\ 0 & -392 & -784 & 1568 \\ 0 & 0 & 0 & 0 \\ 0 & 588 & 1176 & -2352 \end{bmatrix}$

8. $-14^{999} A^T B = \begin{bmatrix} 0 & 14^{999} & 2 \cdot 14^{999} & -4 \cdot 14^{999} \\ 0 & 2 \cdot 14^{999} & 4 \cdot 14^{999} & -8 \cdot 14^{999} \\ 0 & 0 & 0 & 0 \\ 0 & -3 \cdot 14^{999} & -6 \cdot 14^{999} & 12 \cdot 14^{999} \end{bmatrix}$

9. No. $1 \neq 4$.

10. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 5 & 2 & 3 & -1 \\ 1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 2 & 3 & 1 & -1 \\ 1 & -1 & -3 & 0 & -2 \\ 0 & 0 & -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

15. $b = a - 1, c = -1/3, d = 7a - 4, a$ arbitrary

16. We permute the columns of the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & -2 & -1 & -3 \\ 1 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ a \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Next we subtract the first row from the last row, and then add the third row to the first and the second rows with coefficients 1 and 2:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & -5 & -15 \\ 0 & 0 & 1 & -2 & -6 \end{bmatrix} \begin{bmatrix} x \\ a \\ y \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$$

Now we write the answer: $x = 3c - 1, a = 5b + 15c - 4, y = 2b + 6c - 2$, where b, c are arbitrary.

17. We will solve the system for x, d, a. So we rewrite the system (see the solution to Exercise 14 above):

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & -1 & 1 & -1 & -3 & -2 \\ 1 & 0 & 5 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now we subtract the first row from the last one and multiply the second row by -1 :

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 5 & -2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Then we multiply the last row by $1/5$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 1 & -2/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2/5 \end{bmatrix}$$

Finally, we add the last row to the second one:

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 3/5 & 12/5 & 2 \\ 0 & 0 & 1 & -2/5 & -3/5 & 0 \end{bmatrix} \begin{bmatrix} x \\ d \\ a \\ b \\ c \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2/5 \\ -2/5 \end{bmatrix}$$

So our answer is

$x = -2b - 3c + y + 1$,

$d = -0.6b - 2.4c - 2y - 0.4$,

$a = 0.4b + 0.6c - 0.4$

with arbitrary b, c, y.

18. $2A + 3B = [-6, 5, -2, 0, 9, 10]$.

19. $AB^T = 5$

20. $BA^T = 5$

21. $A^T B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 3 & 2 \\ 2 & -1 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 2 & 0 & 0 & 6 & 4 \end{bmatrix}$

22. $B^T A = \begin{bmatrix} 0 & -2 & 2 & 0 & 0 & -4 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 0 & 0 & 6 \\ 0 & 2 & -2 & 0 & 0 & 4 \end{bmatrix}$

23. $(A^T B)^2 = 5A^T B$, and see Answer to 21.

25. $(A^T B)^{1000} = A^T (BA^T)^{999} B = 5^{999} A^T B$, and see Answer to 21.

27. $AB^T = 4$

28. $BA^T = 4$

29. $A^T B = \begin{bmatrix} -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ -1 & 1 & 0 & 3 & 3 & 2 & -1 \\ 1 & -1 & 0 & -3 & -3 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 6 & 6 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

30. $B^T A = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 3 & 3 & -3 & 0 & 0 & 6 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & 0 \\ -1 & -1 & 1 & 0 & 0 & -2 & 0 \end{bmatrix}$

31. $(A^T B)^2 = 4A^T B = \begin{bmatrix} -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ -4 & 4 & 0 & 12 & 12 & 8 & -4 \\ 4 & -4 & 0 & -12 & -12 & -8 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & 8 & 0 & 24 & 24 & 16 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

32. $(A^TB)^3 = 16A^TB$
$$= \begin{bmatrix} -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ -16 & 16 & 0 & 48 & 48 & 32 & -16 \\ 16 & -16 & 0 & -48 & -48 & -32 & 16 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -32 & 32 & 0 & 96 & 96 & 64 & -32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

33. $4^{999}A^TB$

34. $AB^T = \begin{bmatrix} 89/4 & 23 \\ 341/8 & 107/2 \end{bmatrix}$, $BA^T = \begin{bmatrix} 89/4 & 341/8 \\ 23 & 107/2 \end{bmatrix}$,

$A^TB = \begin{bmatrix} 9/2 & 21/4 & 23 \\ 1/2 & 9/4 & -3 \\ 27/2 & 63/4 & 69 \end{bmatrix}$, $B^TA = \begin{bmatrix} 9/2 & 1/2 & 27/2 \\ 21/4 & 9/4 & 63/4 \\ 23 & -3 & 69 \end{bmatrix}$,

$(A^TB)^2 = \begin{bmatrix} 2667/8 & 6363/16 & 6699/4 \\ -297/8 & -633/16 & -809/4 \\ 8001/8 & 19089/16 & 20097/4 \end{bmatrix}$,

$(A^TB)^3 = \begin{bmatrix} 777861/32 & 1857429/64 & 1952517/16 \\ -93351/32 & -222039/64 & -235047/16 \\ 2333583/32 & 5572287/64 & 5857551/16 \end{bmatrix}$.

35. $E_1C = \begin{bmatrix} 3 & 6 & 9 \\ -8 & -10 & -12 \end{bmatrix}$, $E_2C = \begin{bmatrix} 21 & 27 & 33 \\ 4 & 5 & 6 \end{bmatrix}$,

$(E_1)^n = \begin{bmatrix} 3^n & 0 \\ 0 & (-2)^n \end{bmatrix}$, $(E_2)^n = \begin{bmatrix} 1 & 5n \\ 0 & 1 \end{bmatrix}$.

36. $CE_1 = \begin{bmatrix} 2 & 6 & 12 \\ 8 & 15 & 24 \end{bmatrix}$, $CE_2 = \begin{bmatrix} -8 & 2 & 3 \\ -14 & 5 & 6 \end{bmatrix}$,

$DE_1 = \begin{bmatrix} 18 & 24 & 28 \\ 12 & 15 & 16 \\ 6 & 6 & 4 \end{bmatrix}$, $DE_2 = \begin{bmatrix} -12 & 8 & 7 \\ -6 & 5 & 4 \\ 0 & 2 & 1 \end{bmatrix}$,

$E_1E_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -12 & 0 & 4 \end{bmatrix}$, $E_2E_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ -6 & 0 & 4 \end{bmatrix}$,

$(E_1)^n = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}$, $(E_2)^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3n & 0 & 1 \end{bmatrix}$.

37. $\begin{bmatrix} \alpha & 0 \\ 0 & \delta - \gamma\alpha^{-1}\beta \end{bmatrix}$

38. For $n \times n$ diagonal matrices

$$A = \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{bmatrix}, B = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_n \end{bmatrix},$$

we have

$$A + B = \begin{bmatrix} a_1 + b_1 & 0 & \dots & 0 \\ 0 & a_2 + b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n + b_n \end{bmatrix} \text{ and}$$

$$AB = BA = \begin{bmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n b_n \end{bmatrix}.$$

For $m < n$, diagonal $m \times n$ matrices have the form $[A, 0]$ and $[B, 0]$ with $m \times m$ diagonal matrices A, B, and $[A, 0] + [B, 0] = [A + B, 0]$, where 0 is the zero $m \times (n - m)$ matrix. Similarly, sum of diagonal $m \times n$ matrices is diagonal in the case $m > n$.

Any nondiagonal entry of the product of diagonal matrices is the dot product of two rows, each having at most one nonzero entry, and these entries are located at different positions. So the product is a diagonal matrix.

39. Let $A = [a_{ij}], B = [b_{ij}]$ be upper triangular, i.e., $a_{ij} = 0 = b_{ij}$ whenever $i > j$. Then $(A + B)_{ij} = a_{ij} + b_{ij} = 0$ whenever $i > j$, so $A + B$ is upper triangular. For $i > j$, the entry $(AB)_{ij}$ is the product of a row whose first $i - 1 > j$ entries are zero and a column whose entries are zero with possible exception of the fist j entries. So this $(AB)_{ij} = 0$. Thus, AB is upper triangular.

Take upper triangular matrices $A = [1, 2], B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $AB \neq BA$ (they have different sizes). Here is an example with square matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now $AB = B \neq A = BA$.

40. Solution is similar to that of Exercise 39, and these Exercises can be reduced to each other by matrix transposition.

41. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

42. Adding the first column to the third column we obtain the matrix
$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & -2 \\ 2 & 4 & 7 \\ 8 & -2 & 7 \end{bmatrix}.$$
Adding the second column multiplied by 2 to the third column we obtain a lower matrix:
$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 2 & 4 & 15 \\ 8 & -2 & 3 \end{bmatrix}.$$
Now we kill nondiagonal entries in the second, third, and fourth rows using multiples of previous rows. Seven row addition operations bring our matrix to the diagonal matrix
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix}.$$

43. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

44. We kill the entries 5, 5 in the first column by two row addition operations:
$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 16 & 5 & 7 \end{bmatrix}.$$
Adding a multiple of the second row to the last row, we obtain the upper triangular matrix
$$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 11 & 8 & 1 \\ 0 & 0 & -73/11 & 61/11 \end{bmatrix}.$$
Since the diagonal entries are nonzero, we can bring this matrix to its diagonal part
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & -73/11 & 0 \end{bmatrix}$$
by five column addition operations.

§6. Systems of Linear Equations

T1. *Standard description of all solutions for a linear system*

Every x is a solution to the system $Ax = b$ if and only if the augmented matrix $[A, b]$ is the zero matrix. In this case we can write the answer as $0 = 0$. With a big stretch, it is included in the form $z = Cy + d$ on Page 57 with $y = x, C = 0, d = 0, z = 0$. However, it would be better to list it separately to avoid zero-dimensional vector spaces and matrices without rows or columns.

On the other hand, allowing this weird stuff, we con include the outcomes $0 = 1$ and $x = d$ on page 57 into the the outcome $z = Cy + d$ with $y = x$ and $z = x$ respectively.

T2. *Who invented matrices and* $e;$*ementary operations*?

Elementary operations with the augmented matrices were used in China a few centuries before Gauss and Jordan. Even if there are no negative numbers in the initial and final augmented matrices, they may appear on the way. The negative numbers appeared in Europe a few centuries later than in China.

T3. *Triangular matrices*

If we can reduce an m by n matrix A to an upper triangular form U using going-down row addition operations, this means we can write $LA = U$ where L is an m by m lower triangular matrix with ones on the main diagonal. Then $A = L^{-1}U$ where L^{-1} is an m by m lower triangular matrix with ones on the main diagonal. We can find L by doing the going-down row addition operations, with the augmented matrix $[A, 1_m]$.

An invertible n by n matrix A can be written LU with an n by n lower triangular matrix L and an upper triangular n by n matrix U if and only if all northwest corner square submatrices are invertible.

Without restrictions on sizes of U and L, every m by n matrix A can be written in LU-form: $A = [0, 1_m]\begin{bmatrix} 0 \\ A \end{bmatrix}$. Here the sizes of the zero matrices are m by k and k by n with $k = \max(m, n)$.

Any m by n matrix A can be written as $A = LUP$ where U is an m by m lower triangular matrix with ones on the main diagonal, P is n by n permutation matrix, and U is an upper triangular m by n matrix such that the zero diagonal entries go after nonzero diagonal entries.. This is what we do when we solve a system of linear equations with the augmented matrix A.

T4. *Determinant*

Row and column additions operations with a square matrix A do not change its determinant. So to compute the determinant $\det(A)$, we can reduce A to an upper or lower triangular matrix and then multiply the diagonal entries. The matrix A is invertible if and only if $\det(A) \neq 0$. To compute A^{-1} for an n by n matrix A, we can reduce $[A.1_n]$ to a matrix of the form $[1_n, B]$ by invertible row operations. Then $B = A^{-1}$.

T5. *Integer solutions*

Elementary operations allow us to find all integer solutions for a system $Ax = b$ of linear equations with integer coefficients. However we may need column addition operations to do this. To keep track of column operation, we can use the augmented matrix $\begin{bmatrix} A & b \\ 1_n & \end{bmatrix}$. Here is an example: solve the equation $2a + 3b = 7$ in integers a, b. We do column addition operation with integer coefficients:

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & 0 & \\ 0 & 1 & \end{bmatrix} \mapsto \begin{bmatrix} 2 & 1 & 7 \\ 1 & -1 & \\ 0 & 1 & \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 & 7 \\ 3 & -1 & \\ -2 & 1 & \end{bmatrix}$$

In new unknowns, c, d, the equation is $d = 7$. For the original unknowns,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

the answer is

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} c \\ 7 \end{bmatrix} = \begin{bmatrix} 3c - 7 \\ -2c + 7 \end{bmatrix}$$

with an arbitrary c.

T6. *Other linear equations*

Elementary operations can be used sometimes to solve a system of linear equations where the entries of the augmented matrix are functions, polynomials, or differential operators instead of numbers. The difficulty is that some nonzero entries can be not invertible. For example the matrix

$$\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

cannot be reduced to an upper triangular form by addition operation with coefficients being continuous functions of t.

1. $\begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ is invertible; $\det(A) = -4$

3. The matrix is invertible if and only if $abc \neq 0$; $\det(A) = abc$.

5. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is invertible; $\det(A) = 2$

7. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 13/7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ is invertible; $\det(A) = 13$

9. $0 = 1$ (no solutions)

10. We do one row addition operation with the augmented matrix and then drop the zero row:
$-2\begin{matrix}\swarrow\\ \searrow\end{matrix} \begin{bmatrix} 1 & 2 & | & 3 \\ 2 & 4 & | & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix} \mapsto [1, 2 \mid 3]$. Answer: $x = 3 - 2y,\ y$ being arbitrary.

11. $x = -z - 3b + 10,\ y = -z + 2b - 6.$

12. We perform two addition operations on the augmented matrix:
$-1\begin{matrix}\swarrow\\ \searrow\end{matrix} \begin{bmatrix} 1 & 4 & | & 1 \\ 1 & 5 & | & -8 \end{bmatrix} \mapsto -4\begin{matrix}\nearrow\\ \nwarrow\end{matrix} \begin{bmatrix} 1 & 4 & | & 1 \\ 0 & 1 & | & -9 \end{bmatrix} \mapsto$
$\begin{bmatrix} 1 & 0 & | & 37 \\ 0 & 1 & | & -9 \end{bmatrix}.$

Answer: $x = 37, y = -9.$

13. If $t \neq 6 + 2u$, then there are no solutions. Otherwise, $x = -2y + u + 3$, y arbitrary.

15. If $t = 1$, then $x = 1 - y$, y arbitrary.
If $t = -1$, there are no solutions.
If $t \neq \pm 1$, then $x = (t^2 + t + 1)/(t + 1), y = -1/(t + 1).$

17. It is convenient to write the augmented matrix corresponding to the variables y, z, x (rather than x, y, z). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$-5/3\begin{matrix}\swarrow\\ \searrow\end{matrix} \begin{array}{c} \begin{matrix} \;y & \;z & \;x & & \;\; \end{matrix} \\ \begin{bmatrix} 3 & 5 & 2 & | & 2 \\ 5 & 8 & 3 & | & b \end{bmatrix} \end{array} \mapsto$$

$$\begin{matrix} 1/3\cdot \\ -3\cdot \end{matrix} \begin{bmatrix} 3 & 5 & 2 & | & 2 \\ 0 & -1/3 & -1/3 & | & b - 10/3 \end{bmatrix} \mapsto$$

$$-5/3 \begin{array}{l}\nearrow \\ \nwarrow\end{array} \left[\begin{array}{ccc|c} 1 & 5/3 & 2/3 & 2/3 \\ 0 & 1 & 1 & -3b+10 \end{array}\right] \mapsto$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 5b-16 \\ 0 & 1 & 1 & -3b+10 \end{array}\right].$$

Answer: $y = x + 5b - 16, z = -x - 3b + 10,$ x is arbitrary.

18. It is convenient to write the augmented matrix corresponding to the variables x, z, y (rather than x, y, z). So we want to create the identity matrix in the first two columns. This can be achieved by two addition and two multiplication operations:

$$\begin{array}{c} \\ -3/2 \begin{array}{l}\swarrow \\ \searrow\end{array} \end{array} \begin{array}{c} \begin{array}{ccc} x & z & y \end{array} \\ \left[\begin{array}{ccc|c} 2 & 5 & 3 & 2 \\ 3 & 8 & 5 & b \end{array}\right] \end{array} \mapsto$$

$$\begin{array}{r} 1/2\cdot \\ 2\cdot \end{array} \left[\begin{array}{ccc|c} 2 & 5 & 3 & 2 \\ 0 & 1/2 & 1/2 & b-3 \end{array}\right] \mapsto$$

$$-5/2 \begin{array}{l}\nearrow \\ \nwarrow\end{array} \left[\begin{array}{ccc|c} 1 & 5/2 & 3/2 & 1 \\ 0 & 1 & 1 & 2b-6 \end{array}\right] \mapsto$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -5b+16 \\ 0 & 1 & 1 & 2b-6 \end{array}\right].$$

Answer: $x = y - 5b + 16, z = -y + 2b - 6,$ y is arbitrary.

19. No. The halfsum of solutions is a solution.

21. $$A^{-1} = \begin{bmatrix} 7/25 & 4/25 & -1/25 \\ 19/25 & -7/25 & 8/25 \\ -18/25 & 4/25 & -1/25 \end{bmatrix}$$

23. $$A^{-1} = \begin{bmatrix} -3/22 & -1/22 & -41/22 & 3/11 \\ -15/22 & -5/22 & -51/22 & 4/11 \\ 5/22 & 9/22 & 61/22 & -5/11 \\ 15/22 & 5/22 & 73/22 & -4/11 \end{bmatrix}$$

25. If $a \neq 0$, then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d - bc/a \end{bmatrix}.$$

If the first column of A is zero, then $A = 1_2 A = LU$. If the first row of A is zero, then $A = A 1_2 = LU$. Finally, if $a = 0 \neq bc$, then $A \neq LU$.

27. This cannot be done. Suppose $A = LU$. At the position (1,1), we have $0 = A_{11} = L_{11}U_{11}$. Since the first row of A is nonzero, we conclude that $L_{11} \neq 0$. Since the first column of A is nonzero, we conclude that $U_{11} \neq 0$. Thus, $0 = A_{11} = L_{11}U_{11} \neq 0$.

28. This cannot be done, because A is invertible (see the solution of Exercise 5) and $A_{11} = 0$. See the solution of Exercise 27.

29. $A = LU = \begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix}.$

$UL = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & -25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -1 \\ 21 & 33 & 8 \\ -50 & -100 & -25 \end{bmatrix}.$

30. $A = LU$

$= \begin{bmatrix} 1 & -2 & -1 \\ 5 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix}.$

$UL = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 8 \\ 0 & 0 & 13/11 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & 8/11 & 1 \end{bmatrix}$

$= \begin{bmatrix} -11 & -30/11 & -1 \\ 71 & 185/11 & 8 \\ 26/11 & 104/121 & 13/11 \end{bmatrix}.$

32. Answer:
$x = a + b^2 + c^3 - d,$
$y = a + b^2 - 3c^3 + 2d,$
$z = -a - b^2 + 2c^3 - d.$

33. $x = -3(19 + 2d)/8, y = (15 + 2d)/8, z = -(3 + 2d)/8$

35. $x = (15u + 4v)/16, y = (11u + 4v)/16, z = -3u/4$

37. $x = y = 1, z = 0$

39. $x = y = 0, z = 100$

40. It is clear that any nonzero column with at least two entries can be reduced to the first column of the identity matrix by row addition operations. By induction on the number of columns, it follows that any $m \times n$ matrix with linearly independent columns can be reduced by row addition operations to the matrix of the first n columns of 1_m provided that $m > n$. Therefore any invertible $m \times m$ matrix can be reduced by row addition operations to the diagonal matrix with the first $m - 1$ diagonal entries being ones, and the last entry being the determinant. One row multiplication operation applied to this matrix gives 1_m.

Therefore multiplication by an invertible matrix on the left is equivalent to performing row addition operations and a row multiplication operation.

If $A = 0$, then $b = 0, A' = 0.$ and $b' = 0$, so there is nothing to prove. Similarly, the case $A' = 0$ is trivial. Assume now that $A \neq 0$ and $A' \neq 0$.

Let B be the submatrix in $[A, b]$ such that the rows of B form a basis for the row space of A, and let B' be a similar matrix for $[A', b']$. By Theorem 6.11, $B' = DB$ and $B = D'B'$ for some matrices D, D'. We have $B = D'DB$ and $B' = DD'$. Since the rows of B are linearly independent, $D'D$ is the identity matrix. Since the rows of B' are linearly independent, DD' is the identity matrix. So D is invertible, hence B, B' have the same size. So row operations on A allows us to change the rows of B to the rows of B'. Now by row addition operations we can make the other rows of $[A, b]$ (if any) equal to remaining rows of $[A', b]'$ (if any). A row permutation operation finishes the job.

41. We use the parts of the previous solution. In particular, it is clear that the rank of the matrices of $[A, b]$ and $[A', b']$ are the same. By row addition operations we can make the last $m - m'$ rows of $[A, b]$ to be zeros. Then, as shown in the previous solution, we can manipulate the first m' rows to be the rows of $[A', b']$ by row addition operations and a row multiplication operation.

The only thing remaining to show is how to replace a row multiplication operation by row addition operations in the presence of a zero row. Here is how this can be done:

$$1 \searrow \begin{bmatrix} r \\ 0 \end{bmatrix} \mapsto (d-1) \nearrow \begin{bmatrix} r \\ r \end{bmatrix} \mapsto -1/d \searrow \begin{bmatrix} dr \\ r \end{bmatrix} \mapsto \begin{bmatrix} dr \\ 0 \end{bmatrix},$$

where multiplication of a row r by a nonzero number d is accomplished by three addition operations.

Chapter 3. Tableaux and Pivoting

§7. Standard and Canonical Forms for Linear Programs

1. Set $u = y + 1 \geq 0$. Then $f = 2x + 3y = 2x + 3u - 3$ and $x + y = x + u - 1$. A canonical form is

$$-f = -2x - 3u + 3 \to \min, x + u \leq 6, u, x \geq 0.$$

A standard form is

$$-f = -2x - 3u + 3 \to \min, x + u + v = 6, u, v, x \geq 0$$

with a slack variable $v = 6 - x - u \geq 0$.

2. Excluding $y = x+1$ and using $y \geq 1$, we obtain the canonical form

$$-x \to \min,\ 2x \leq 8, x \geq 0.$$

Introducing a slack variable $z = 8 - 2x$, we obtain the standard form

$$-x \to \min,\ 2x + z = 8, x \geq 0, z \geq 0.$$

3. We solve the equation for x_3:

$x_3 = 3 - 2x_2 - 3x_4$

and exclude x_3 from the LP:

$$x_1 - 7x_2 + 3 \to \min,\ x_1 - x_2 + 3x_4 \geq 3, \text{ all } x_i \geq 0.$$

A canonical form is

$$x_1 - 7x_2 + 3 \to \min,\ -x_1 + x_2 - 3x_4 \leq -3,\ \text{ all } x_i \geq 0.$$

A standard form is

$$x_1 - 7x_2 + 3 \to \min,\ -x_1 + x_2 - 3x_4 + x_5 = -3,\ \text{ all } x_i \geq 0$$

with a slack variable $x_5 = x_1 - x_2 + 3x_4 - 3$.

5. Set $t = x + 1 \geq 0, u = y - 2 \geq 0$. The objective function is $x + y + z = t + u + z + 1$. Then a standard and a canonical form for our problem is

$$x + u + z + 1 \to \min; t, u, z \geq 0.$$

6. This mathematical program has exactly two optimal solution, but the set of optimal solutions of any LP is convex and hence cannot consist of exactly two optimal solutions (cf. Exercise 19 in §6.). Each of two optimal solution can be the optimality region for a linear program. For example, min $= -26$ at $x = 1, y = -3, z = 0$ is the only answer for the linear program $-26 \to \min, x = 1, y = -3, z = 0$.

7. Using standard tricks, a canonical form is

$$-x \to \min, x \le 3, -x \le -2, x \ge 0.$$

A standard form is

$$-x \to \min, x + u = 3, -x + v = -2; x, u, v \ge 0$$

with two slack variables.

8. Excluding $y = 1 - x$ from the LP, we obtain

$$f = -x + z + 2 \to \max, z \ge 0.$$

Writing $x = u - v$ and replacing f by $-f$, we obtain a normal and standard form:

$$-f = u - v - z - 2 \to \min; u, v, z \ge 0.$$

It is clear that the program is unbounded.

9. One of the given equations reads
$-5 - x - z = 0,$
which is inconsistent with given constraints $x, z \ge 0$. So we can write very short canonical and standard forms:
$0 \to \min, 0 \le -1; x, y, z \ge 0$ and $0 \to \min, 0 = 1; x, y, z \ge 0$.

10. The first matrix product is not defined.

11. Set $x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}]^T$ and $c = [3, -1, 1, 3, 1, -5, 1, 3, 1]$. Using standard tricks, we obtain the canonical form
$cx \to \min, Ax \le b, x \ge 0$
with

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & -2 & 3 & 1 \\ 2 & -2 & -2 & 2 & 3 & -1 & -2 & 1 & 1 \\ -2 & 2 & 2 & -2 & -3 & 1 & 2 & -1 & -1 \\ 1 & 0 & 0 & 0 & 3 & -1 & -2 & 0 & -1 \\ -1 & 0 & 0 & 0 & -3 & 1 & 2 & 0 & 1 \end{bmatrix}$$

and $b = [-3, -1, 2, -2, 0, 0]^T$.

Excluding a couple of variables using the two given equations, we would get a canonical form with two variables and two constraints less. A standard form can be obtained from the canonical form by introducing a column u of slack variables:

$$cx \to \min, Ax + u = b, x \ge 0, u \ge 0.$$

§8. Pivoting Tableaux

1.
$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & & & \\
\wedge\!| & \wedge\!| & \wedge\!| & \wedge\!| & \wedge\!| & & & \\
a & b & c & d & e & 1 & & \\
\end{array}$$
$$\begin{bmatrix} .3 & 1.2 & .7 & 3.5 & 5.5 & -50 \\ 73 & 96 & 20253 & 890 & 279 & -4000 \\ 9.6 & 7 & 19 & 57 & 22 & -1000 \\ 10 & 15 & 5 & 60 & 8 & 0 \end{bmatrix} \begin{array}{l} = u_1 \geq 0 \\ = u_2 \geq 0 \\ = u_3 \geq 0 \\ = C \to \min \end{array}$$

3. $A = \begin{bmatrix} 3 & -1 & 2 & 2 \\ -1 & 0 & 0 & 2 \\ -1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \end{bmatrix}$

5. Canonical form:
$y - 5z + 2 \to \min$,
$-3x - y + 5z \leq 3$,
$-x - y \leq -10$,
$x + y \leq 10$,
$-2y + 10z \leq -7$;
$x, y, z \geq 0$.
Standard form:
$y - 5z + 2 \to \min$,
$-3x - y + 5z + u = 3$,
$x + y = 10$,
$-2y + 10z + v = -7$;
$x, y, z, u, v \geq 0$.

7. Canonical form: $-x + z + 2 \to \min$,
$-3x - 2y - z \leq 2, x - 3y \leq 1, 2y - 2z \leq 0; x, y, z \geq 0$.
Standard form: $-x + z + 2 \to \min$,
$-3x - 2y - z + u = 2, x - 3y + v = 1, 2y - 2z + w = 0$;
$x, y, z, u, v, w \geq 0$.

9. The matrix is not square.

11. $$\begin{array}{cccc} z & a & 3 & x \end{array}$$
$$\begin{bmatrix} -1 & 2 & b+3 & a+1 \\ -1 & 2 & 3 & 1 \end{bmatrix} \begin{array}{l} = y \\ = 1 \end{array}$$

12. $$\begin{array}{cccc|l} 1 & a & 3 & z & \\ \end{array}$$
$$\begin{bmatrix} 1+a & -2a & b-3a & a \\ 1 & -2 & -3 & 1 \end{bmatrix} \begin{matrix} = y \\ = x \end{matrix}$$

13. $$\begin{array}{c} 2 \\ \begin{bmatrix} 1/5 \end{bmatrix} \end{array} \; = x$$

14. $$\begin{array}{ccccc} 1 & a & 0 & x & x \end{array}$$
$$\begin{bmatrix} 1 & 0 & b & a & -3 \\ -1 & 2^* & 3 & 1 & 0 \end{bmatrix} \begin{matrix} = y \\ = z \end{matrix} \quad \mapsto$$
$$\begin{array}{ccccc} 1 & z & 0 & x & x \end{array}$$
$$\begin{bmatrix} 1 & 0 & b & a & -3 \\ 1/2 & 1/2 & -3/2 & -1/2 & 0 \end{bmatrix} \begin{matrix} = y \\ = a \end{matrix}$$

15. $$\begin{array}{ccccc} 1 & u & 0 & x & 1 \end{array}$$
$$\begin{bmatrix} 1 & 0 & b & a & -3 \\ 0 & 1 & 0 & 0 & -1 \\ 1/2 & 1/2 & -3/2 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{matrix} = y \\ = z \\ = 0 \\ = v \end{matrix}$$

16. $$\begin{array}{ccccc} x_1 & x_2 & x_8 & x_4 & 1 \end{array}$$
$$\begin{bmatrix} 4/3 & -2/3 & 1/3 & 2/3 & -3 \\ -1/3 & -4/3 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1/3 & -2/3 & 1/3 & -1/3 & 0 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{matrix} = x_5 \\ = x_6 \\ = x_7 \\ = x_3 \\ = v \end{matrix}$$

17. $$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_7 & x_6 & 1 \end{array}$$
$$\begin{bmatrix} 1/3 & 0 & 1/3 & 1/3 & -1/3 & 1/3 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -2/3 & 2 & 10/3 & 4/3 & -1/3 & 1/3 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -2/3 & 1 & 1/3 & 4/3 & -1/3 & 7/3 & 3 \end{bmatrix} \begin{matrix} = x_5 \\ = x_8 \\ = x_9 \\ = x_{10} \\ = v \end{matrix}$$

18. Let us show that every column b of A equals to the corresponding column b' of A'. We set the corresponding variable on the top to be 0, and the other variables on the top to be zeros. Then the variables on the side take certain values, namely, $y = b = b'$.

§9. Standard Row Tableaux

1. Passing from the standard row tableau on page 95 to the canonical form (i.e., dropping the slack variables), we obtain a Linear program with one variable: $y/2 - 15/2 \to \min$,

$16y - 26 \geq 0, -3y/2 + 15/2 \geq 0, 3y/2 - 15/2 \geq 0, y \geq 0$.

We rewrite our constraints: $y \geq 0, 13/8, 5; y \leq 5$, so $y = 5$. In terms of the standard tableau, our answer is

$\min(-z) = -5$ at $y = 5, w_1 = 54, w_2 = w_3 = 0$.

In terms of the original variables, our answer is

$\max(z) = 5$ at $u = -3, v = -4, x = -1, y = 5$.

2. $$\begin{array}{c} \begin{array}{ccc} x & y & 1 \end{array} \\ \begin{bmatrix} -4 & -5 & 7 \\ -2 & -3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ = u \\ = -P \to \min \end{array}$$

with a slack variable $u = 7 - 4x - 5y$

3. $$\begin{array}{c} \begin{array}{ccccc} x' & x'' & y' & y'' & 1 \end{array} \\ \begin{bmatrix} -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = -x \to \min \end{array}$$

with $x = x' - x'', y = y' - y''$ and slack variables u_i.

5. We multiply the last row by -1 and remove the second and third rows from the tableau

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 & -1 & -2 & -3 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = x_2 \\ = -v \to \min \end{array}$$

$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 3x_3 + x_4 + x_5$.

The tableau is not standard because x_2 occurs twice. We pivot on 1 in the x_2-row and x_6-column:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & -3 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1^* & 1 \\ 1 & -1 & 0 & -1 & -1 & -2 & -3 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = x_2 \\ = -v \to \min \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_2 & 1 \end{array} \\ \begin{bmatrix} 1 & -2 & -2 & 0 & -3 & 1 & -1 \\ 0 & -2 & -3 & -1 & 0 & 1 & -1 \\ 1 & 3 & 6 & 1 & -1 & -2 & -1 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = x_6 \\ = -v \to \min \end{array} .$$

Now we combine two x_2 from the top and obtain the standard tableau

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{array} \\ \begin{bmatrix} 1 & -1 & -2 & 0 & -3 & -1 \\ 0 & -1 & -3 & -1 & 0 & -1 \\ 1 & 1 & 6 & 1 & -1 & -1 \end{bmatrix} \end{array} \begin{array}{r} = x_7 \\ = x_6 \\ = -v \to \min \end{array} .$$

The equations

$x_8 = -x_1 + 2x_3 + x_4 + x_6, x_9 = -x_1 + 2x_2 + 5x_3 + x_4 + x_5$
relate this LP with the original LP. After we solve the program without x_8 and x_9, we complete the answer with the values for x_8 and x_9. By the way, looking at the equation for x_6 (the second row of the standard tableau), we see that the program is infeasible.

7. We pivot on the first 1 in the first row and then on 3 in the second row:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{r} = x_2 \\ = x_4 \\ = v \to \min \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{ccccccc} x_2 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & 3^* & 2 & -3 & 2 & -2 \\ -1 & 2 & 4 & 2 & -2 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{r} = x_1 \\ = x_4 \\ = v \to \min \end{array} \mapsto$$

$$\begin{array}{c} \begin{array}{ccccccc} x_2 & x_2 & x_4 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 2/3 & 0 & -1/3 & -1/3 & 2 & -1/3 & -2/3 \\ 1/3 & 0 & 1/3 & -2/3 & 1 & -2/3 & 2/3 \\ 1/3 & 2 & 4/3 & -2/3 & 2 & -5/3 & 8/3 \end{bmatrix} \end{array} \begin{array}{r} = x_1 \\ = x_3 \\ = v \to \min \end{array} .$$

Now we combine two x_2-columns and two x_4-columns and obtain the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_2 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 2/3 & -2/3 & 2 & -1/3 & -2/3 \\ 1/3 & -1/3 & 1 & -2/3 & 2/3 \\ 7/3 & 2/3 & 2 & -5/3 & 8/3 \end{bmatrix} \end{array} \begin{array}{r} = x_1 \\ = x_3 \\ = v \to \min \end{array} .$$

Do not forget the constraints $x_7, x_8, x_9, x_{10} \geq 0$ outside the tableaux.

T1. *Several objective functions*?

When x is a row or column, $x \geq 0$ means that every entry is ≥ 0. Notice that both $x \geq 0$ and $-x \geq 0$ can be false. Here is an example: $x = [1, -1]$. So it is not clear what optimization of two objective functions at once means.

9. We pivot the three zeros from the right margin to the top and drop the corresponding columns:

$$\begin{array}{c} \begin{array}{ccccccc} x_7 & x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 1^* & 0 & 1 & 1 & -3 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 1 & -2 \\ -1 & 2 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 2 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ = 0 \\ = 0 \\ = 0 \\ = x_1 \\ = v \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{cccccc} x_2 & x_3 & x_4 & x_5 & x_6 & 1 \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 3 & 2 & -3 & 2 & -2 \\ 2 & 4 & 2 & -2 & 1^* & 0 \\ 2 & 4 & 2 & -3 & 2 & 1 \\ 1 & 1 & 2 & -2 & 3 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = 0 \\ = 0 \\ = x_1 \\ = v \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{ccccc} x_2 & x_3 & x_4 & x_5 & 1 \end{array} \\ \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ -4 & -5 & -2 & 1^* & -2 \\ -2 & -4 & -2 & 2 & 0 \\ -2 & -4 & -2 & 1 & 1 \\ -5 & -11 & -4 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = 0 \\ = x_6 \\ = x_1 \\ = v \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 6 & 8 & 3 & 2 \\ 4 & 5 & 2 & 2 \\ 6 & 6 & 2 & 4 \\ 2 & 1 & 0 & 3 \\ 11 & 9 & 4 & 11 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_7 \\ = x_5 \\ = x_6 \\ = x_1 \\ = v \to \min . \end{array}$$

Now we take the first equation

$$x_7 = 6x_2 + 8x_3 + 3x_4 + 2$$

outside the table (since x_7 is not required to be ≥ 0) and obtain the standard tableau

$$\begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 4 & 5 & 2 & 2 \\ 6 & 6 & 2 & 4 \\ 2 & 1 & 0 & 3 \\ 11 & 9 & 4 & 11 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_5 \\ = x_6 \\ = x_1 \\ = v \to \min\,. \end{array}$$

T2. *Constraints in sign*

Remember that all decision variables in a standard tableau are required to be non-negative even when it is not written on side. For example, $u \geq 0$ in the second tableau on page 97 because the tableau is called standard.

On the other hand, the decision variable (if it is present) is not restricted in sign. Also its name need not be distinct from those for the decision variables.

T3. ∞ *and* $-\infty$

They are not numbers in our class. The reason is that there is no good definition for their sum, while we want the sum of any two numbers to be defined.

Chapter 4. Simplex Method

Scheme of simplex method

Simplex Method

All tableaux are standard.

All tableaux are feasible in Phase 2.

$$\begin{bmatrix}\text{initial}\\ \text{tableau}\end{bmatrix} \overset{\text{Phase 1}}{\underset{\textit{pivot steps}}{--\rightarrow}} \begin{cases} \nearrow \begin{bmatrix}\text{feasible tableau}\\ \text{terminal for Phase 1}\\ \text{initial for Phase 2}\end{bmatrix} : \quad \begin{array}{cc} \oplus & 1 \end{array} \begin{bmatrix} & \oplus \\ & \end{bmatrix} \begin{array}{l} = \oplus \\ \to \min \end{array} \quad \overset{\text{Phase 2}}{\underset{\textit{pivot steps}}{--\rightarrow}} \\ \searrow \begin{array}{c} \oplus \; 1 \\ [\ominus -] = \oplus \quad \textit{bad row} \\ \text{a terminal tableau} \\ \text{LP is infeasible} \end{array} \end{cases}$$

$$\overset{\text{Phase 2}}{\underset{\textit{pivot steps}}{----\rightarrow}} \begin{cases} \nearrow \begin{bmatrix}\text{optimal}\\ \text{tableau}\end{bmatrix} \begin{array}{c}\text{(terminal)}\\ b \ge 0, c \ge 0\end{array} : \begin{bmatrix} A & \oplus \\ \oplus & d \end{bmatrix} \\ \qquad \begin{array}{cc} x & 1 \end{array} \\ \qquad \begin{bmatrix} A & b \\ c & d \end{bmatrix} \begin{array}{l} = y \\ \to \min \end{array} \\ \qquad \min = d \text{ at } x = 0, y = b \\ \searrow \text{bad column in Phase 2 :} \\ \qquad \begin{array}{c} \oplus \end{array} \\ \qquad \begin{bmatrix} \oplus \\ - \end{bmatrix} \begin{array}{l} = \oplus \\ \to \min \end{array} \\ \qquad \text{a terminal tableau} \\ \qquad \min = -\infty \end{cases}$$

§10. Simplex Method, Phase 2

T1. *Redundant constraints*

Sometimes it is clear that a constraint in a linear program is redundant, that is, it follows from the other constraints, that is, dropping it does not change the feasible region.

Example 1. Suppose that a linear program contains the following three equations:

$$x + y + z = 1, 2x + 3y + z = 3, 3x + 4y + 2z = 4.$$

Then any one of them is redundant and can be dropped.

Example 2. Suppose that a linear program in canonical form contains the following constraints:

$$-x - 2y \le 1, x \ge 0, y \ge 0, z \ge 0.$$

Then the first constraint is redundant.

In a standard tableau, this constraint is represented by the row

$$\begin{array}{c} \begin{array}{cccc} x & y & z & 1 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} \end{array} = *\,.$$

This row can be dropped.

T2. *Redundant variables*

Sometimes we know the range of a variable in every optimal solution. Then we can exclude this variable from the standard tableau.

Example 3. In the standard tableau

$$\begin{array}{c} \begin{array}{cccc} x & y & z & 1 \end{array} \\ \begin{bmatrix} * & -2 & * & * \\ * & 0 & * & * \\ * & -1 & * & * \\ * & -1 & * & * \\ * & 0 & * & * \\ * & 3 & * & * \end{bmatrix} \end{array} \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ \to \min \end{array}$$

the y-column can be dropped because y must be 0 in every optimal solution.

Example 4. In the standard tableau

$$\begin{array}{cl} \begin{array}{cccc} x & y & z & 1 \end{array} & \\ \begin{bmatrix} * & -2 & * & * \\ * & 0 & * & * \\ * & -1 & * & * \\ * & 0 & * & * \end{bmatrix} & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ \to \min \end{array} \end{array}$$

the y-column can be dropped because we can replace the value for y in each optimal solution by 0 keeping it optimal.

Example 5. In the standard tableau

$$\begin{array}{cl} \begin{array}{cccc} x & y & z & 1 \end{array} & \\ \begin{bmatrix} * & * & * & * \\ -1 & 0 & -2 & 0 \\ * & * & * & * \end{bmatrix} & \begin{array}{l} = u_1 \\ = u_2 \\ \to \min \end{array} \end{array}$$

the x- and z-columns can be dropped because the u_2-row forces $x = z = 0$ in every feasible solution.

T3. *Small standard tableaux*

In simplex method, we cannot pivot standard tableaux which have only one column or only one row. Such tableaux are terminal. For example, the standard tableau

$$\begin{array}{cl} \begin{array}{cccc} x & y & z & 1 \end{array} & \\ \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} & \to \min \end{array}$$

is optimal, the standard tableau

$$\begin{array}{cl} \begin{array}{cccc} x & y & z & 1 \end{array} & \\ \begin{bmatrix} 1 & -2 & 0 & 1 \end{bmatrix} & \to \min \end{array}$$

is feasible with a bad column, the standard tableau

$$\begin{array}{cl} 1 & \\ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ \to \min \end{array} \end{array}$$

is optimal, and the standard tableau

$$\begin{array}{cl} 1 & \\ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix} & \begin{array}{l} = u_1 \\ = u_2 \\ = u_3 \\ \to \min \end{array} \end{array} \quad \text{has a bad row.}$$

T4. *Example of cycling*
See
http://www.math.ubc.ca/˜ anstee/math340/cyclingLP.pdf
for an example of cycling in Phase 2. In this example, there are 4 variables on top and 3 at side. The objective function z is maximized, so to multiply the last row by -1 to obtain a minimization problem. The cycle consists of 6 tableaux. Bland's rule allows us to get an optimal tableau in 8 pivot steps.

1. The tableau is optimal, so the basic solution is optimal:
min $= 0$ at $a = b = c = d = 0$,
$y_1 = 0.4, y_2 = 0.4, y_3 = 0, y_4 = 0.5, y_5 = 1, y_6 = 0.1$.

2. The y_2-row is bad. The program is infeasible.

3. The first column is bad. However since the tableau is not feasible, this is not sufficient to conclude that the program is unbounded. Still we set $z_2 = z_3 = z_4 = 0$, and see what happens as $z_1 \to \infty$. We have $y_1 = 0.4 \geq 0, y_2 = 3z_1 + 0.4 \geq 0, y_3 = 0.6z_1 \geq 0, y_4 = 0.6z_1 + 0.5 \geq 0, y_5 = 0.1z_1 - 0.1 \geq 0, y_6 = 0.1 \geq 0$ for $z_1 \geq 1$, and the objective function $-11z_1 \to -\infty$. So min $= -\infty$.

4. False. The converse is true.

5. True

6. True

7. First we write the program in a standard tableau and then we apply the simplex method (Phase 2):

$$\begin{array}{c}
\begin{array}{cccc} x_1 & x_2 & x_3 & 1 \end{array} \\
\begin{bmatrix} -40^* & -20 & -60 & 1200 \\ -4 & -1 & -6 & 300 \\ -0.2 & -0.7 & -2 & 40 \\ -100 & -100 & -800 & 8000 \\ -0.1 & -0.3 & -0.8 & 8 \\ -2 & -3 & -7 & 0 \end{bmatrix}
\end{array}
\begin{array}{l} \\ = u_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c}
\begin{array}{cccc} u_1 & x_2 & x_3 & 1 \end{array} \\
\begin{bmatrix} -0.025 & -0.5 & -1.5 & 30 \\ 0.1 & 1 & 0 & 180 \\ 0.005 & -0.6 & -1.7 & 34 \\ 2.5 & -50 & -650 & 5000 \\ 0.0025 & -0.25^* & -0.65 & 5 \\ 0.05 & -2 & -4 & -60 \end{bmatrix}
\end{array}
\begin{array}{l} \\ = x_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = u_5 \\ = -P \to \min \end{array} \quad \mapsto$$

$$\begin{array}{cccc|l} u_1 & u_5 & x_3 & 1 & \\ \end{array}$$
$$\left[\begin{array}{cccc} -0.03 & 2 & -0.2 & 20 \\ 0.11 & -4 & -2.6 & 200 \\ 0.001 & 2.4 & -0.14 & 22 \\ 2 & 200 & -520 & 4000 \\ 0.01 & -4 & -2.6 & 20 \\ 0.03 & 8 & 1.2 & -100 \end{array}\right] \begin{array}{l} = x_1 \\ = u_2 \\ = u_3 \\ = u_4 \\ = x_2 \\ = -P \to \min. \end{array}$$

This tableau is optimal, so

$$\max(P) = 100 \text{ at } x_1 = 20, x_2 = 20, x_3 = 0.$$

The zero values for the nonbasic slack variables u_1 and u_5 indicate that the corresponding resource limits are completely used (no slack there). The other resources are not completely used; some reserves left.

9. First we solve the system of linear equations for a, b and the objective function f and hence obtain the standard tableau

$$\begin{array}{ccc} c & d & 1 \end{array}$$
$$\left[\begin{array}{ccc} 1 & 2 & -0.5 \\ -2 & -3 & 1.5 \\ 0.1 & 0.1 & 1.5 \end{array}\right] \begin{array}{l} = a \\ = b \\ = f \to \min. \end{array}$$

The tableau is not row feasible so we cannot apply Phase 2. Until we learn Phase 1, we can pivot at random:

$$\begin{array}{ccc} c & d & 1 \end{array}$$
$$\left[\begin{array}{ccc} 1^* & 2 & -0.5 \\ -2 & -3 & 1.5 \\ 0.1 & 0.1 & 1.5 \end{array}\right] \begin{array}{l} = a \\ = b \\ = f \to \min \end{array} \quad \mapsto$$

$$\begin{array}{ccc} a & d & 1 \end{array}$$
$$\left[\begin{array}{ccc} 1 & -2 & 0.5 \\ -2 & 1 & 0.5 \\ 0.1 & -0.1 & 1.55 \end{array}\right] \begin{array}{l} = c \\ = b \\ = f \to \min. \end{array}$$

Now the tableau is feasible, and we can use Phase 2:

$$\begin{array}{ccc} a & d & 1 \end{array}$$
$$\left[\begin{array}{ccc} 1 & -2^* & 0.5 \\ -2 & 1 & 0.5 \\ 0.1 & -0.1 & 1.55 \end{array}\right] \begin{array}{l} = c \\ = b \\ = f \to \min \end{array} \quad \mapsto$$

$$\begin{array}{ccc} a & c & 1 \end{array}$$
$$\left[\begin{array}{ccc} 0.5 & -0.5 & 0.25 \\ -1.5 & -0.5 & 0.75 \\ 0.05 & 0.05 & 1.525 \end{array}\right] \begin{array}{l} = d \\ = b \\ = f \to \min. \end{array}$$

The tableau is optimal, so
min $= 1.525$ at $a = 0, b = 0.75, c = 0, d = 0.25$.

10. First we write the program in the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 1 & 3 \\ 1 & -1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & -2 \end{bmatrix} \end{array} \begin{array}{l} \\ = x_5 \\ = x_6 \\ \to \min \end{array}$$

The tableau is optimal, so min $= -2$ at $x_1 = x_2 = x_3 = x_4 = 0, x_5 = 3, x_6 = 1$.

11. Set $f = x_2 + 2x_3 - 2$ (the objective function). We write the program in the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 3 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 & 2 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_5 \\ = x_6 \\ = -f \to \min \end{array}$$

The tableau is feasible, and two columns are bad (namely, the x_2-column and x_3-column), so the program is unbounded ($\max(f) = \infty$).

12. Set $f = x_2 + 2x_3 + 2$ (the objective function to maximize). We write the program in the standard tableau

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & -3 \\ 1 & 1 & 2 & 1 & -1 \\ 0 & -1 & -2 & 0 & -2 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_5 \\ = x_6 \\ = -f \to \min \end{array}$$

We set $x_1 = 3$ and obtain a feasible tableau

$$\begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_4 & 1 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ -1 & -2 & 0 & -2 \end{bmatrix} \end{array} \begin{array}{r} \\ = x_5 \\ = x_6 \\ = -f \to \min \end{array}$$

with the first (and the second) column bad. So the program is unbounded, therefore the original program is unbounded.

13. If the row without the last entry is nonnegative, then the tableau is optimal; else the LP is unbounded.

§11. Simplex Method, Phase 1

1. The second row (v-row) is bad, so the LP is infeasible.

2. The tableau is optimal, so the basic solution is optimal:
min $= 0$ at $x = y = z = 0, u = 2, v = 0$.
This is the only optimal solution.

3. This is a feasible tableau with a bad column (the z-column). So the LP is unbounded (z and hence w can be arbitrarily large).

5. The tableau is standard. According to the simplex method, we pivot on 1 in the first row:

$$\begin{array}{cc} \begin{array}{cccc} a & b & c & 1 \end{array} & \\ \begin{bmatrix} 1^* & 2 & 3 & -1 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{l} = d \\ = e \\ \to \min \end{array} \end{array} \quad \mapsto$$

$$\begin{array}{cc} \begin{array}{cccc} d & b & c & 1 \end{array} & \\ \begin{bmatrix} 1 & -2 & -3 & 1 \\ 2 & -4 & -5 & 5 \\ -1 & 3 & 3 & -1 \end{bmatrix} & \begin{array}{l} = a \\ = e \\ \to \min \end{array} \end{array}$$

The tableau is feasible, and the d-column is bad, so the program is unbounded (min $= -\infty$).

7. We scale the last column and then pivot on the first 1 in the first row to get both c on the top:

$$\begin{array}{cc} \begin{array}{cccc} a & b & c & 1 \end{array} & \\ \begin{bmatrix} 1^* & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 3 & 0 \end{bmatrix} & \begin{array}{l} = c \\ = d \\ = f \to \min \end{array} \end{array} \quad \mapsto$$

$$\begin{array}{cc} \begin{array}{cccc} c & b & c & 1 \end{array} & \\ \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 2 & -1 \end{bmatrix} & \begin{array}{l} = a \\ = d \\ = f \to \min . \end{array} \end{array}$$

Now we combine two c-columns and obtain the standard tableau

$$\begin{array}{cc} \begin{array}{ccc} b & c & 1 \end{array} & \\ \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} & \begin{array}{l} = a \\ = d \\ = f \to \min . \end{array} \end{array}$$

The first row is bad, so the program is infeasible. In fact, the first constraint in the original tableau is inconsistent with the constraint $a \geq 0$.

9. True

10. False

11. We use the simplex method:

$$\begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & 1 \end{array}\\ \left[\begin{array}{ccccc} 1^* & 0 & -2 & -3 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 & 3 \\ 1 & -1 & 1 & 0 & 2 \end{array}\right]\end{array} \begin{array}{l} \\ = x_5 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c}\begin{array}{ccccc} x_5 & x_2 & x_3 & x_4 & 1 \end{array}\\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 3 & 1 \\ -1 & 1 & -1 & -2 & 0 \\ 2 & -1^* & 4 & 7 & 5 \\ 1 & -1 & 3 & 3 & 3 \end{array}\right]\end{array} \begin{array}{l} \\ = x_1 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c}\begin{array}{ccccc} x_5 & x_7 & x_3 & x_4 & 1 \end{array}\\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 3 & 1 \\ 1 & -1 & 3 & 5 & 5 \\ 2 & -1 & 4 & 7 & 5 \\ -1 & 1 & -1 & -4 & -2 \end{array}\right]\end{array} \begin{array}{l} \\ = x_1 \\ = x_6 \\ = x_2 \\ \to \min . \end{array}$$

Phase 1 was done in one pivot step, and Phase 2 also was done in one pivot step, because we obtain a feasible tableau with a bad column (x_5-column). The program is unbounded.

12. We use the simplex method:

$$\begin{array}{c}\begin{array}{cccc} x_1 & x_2 & x_3 & 1 \end{array}\\ \left[\begin{array}{cccc} 1^* & 0 & -1 & -1 \\ -1 & 3 & 1 & 0 \\ 3 & -1 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]\end{array} \begin{array}{l} \\ = x_4 \\ = x_5 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c}\begin{array}{cccc} x_4 & x_2 & x_3 & 1 \end{array}\\ \left[\begin{array}{cccc} 1 & 0 & 1 & 1 \\ -1 & 3^* & 0 & -1 \\ 3 & -1 & 5 & 4 \\ 1 & -1 & 2 & 1 \\ 1 & -1 & 0 & 1 \end{array}\right]\end{array} \begin{array}{l} \\ = x_1 \\ = x_5 \\ = x_6 \\ = x_7 \\ \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c}
\begin{array}{cccc} x_4 & x_5 & x_3 & 1 \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1/3 & 1/3 & 0 & 1/3 \\
8/3 & -1/3 & 5 & 11/3 \\
2/3 & -1/3^* & 2 & 2/3 \\
2/3 & -1/3 & 0 & 2/3
\end{bmatrix}
\end{array}
\begin{array}{l} = x_1 \\ = x_2 \\ = x_6 \\ = x_7 \\ \to \min \end{array}
\quad \mapsto$$

$$\begin{array}{c}
\begin{array}{cccc} x_4 & x_7 & x_3 & 1 \end{array} \\
\begin{bmatrix}
1 & 0 & 1 & 1 \\
1 & -1 & 2 & 1 \\
6/3 & 1 & 3 & 9/3 \\
2 & -3 & 6 & 2 \\
0 & 1 & -2 & 0
\end{bmatrix}
\end{array}
\begin{array}{l} = x_1 \\ = x_2 \\ = x_6 \\ = x_5 \\ \to \min . \end{array}$$

The x_3-column is bad, so the program is unbounded.

13. We use the simplex method:

$$\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{array} \\
\begin{bmatrix}
1 & 0 & -1 & -1 & 0 & 1 \\
-1 & 3 & 1^* & 0 & -2 & -1 \\
3 & -1 & 2 & 1 & 2 & 1 \\
1 & -1 & 1 & 0 & -1 & 1 \\
0 & -1 & -1 & 0 & -1 & 2
\end{bmatrix}
\end{array}
\begin{array}{l} = x_6 \\ = x_7 \\ = x_8 \\ = x_9 \\ \to \min \end{array}
\quad \mapsto$$

$$\begin{array}{c}
\begin{array}{cccccc} x_1 & x_2 & x_7 & x_4 & x_5 & 1 \end{array} \\
\begin{bmatrix}
0 & 3 & -1 & -1 & -2 & 0 \\
1 & -3 & 1^* & 0 & 2 & 1 \\
5 & -7 & 2 & 1 & 6 & 3 \\
2 & -4 & 1 & 0 & 1 & 2 \\
-1 & 2 & -1 & 0 & -3 & 1
\end{bmatrix}
\end{array}
\begin{array}{l} = x_6 \\ = x_3 \\ = x_8 \\ = x_9 \\ \to \min . \end{array}$$

The fist column in this feasible tableau is bad, so the program is unbounded.

§12. Geometric Interpretation

T1. *Number of corners and pivot steps*

Given $(m+1) \times (n+1)$ standard tableau (so we have $m+n$ decision variables), there are at most $(m+n)!$ standard tableaux we can get from the initial tableau by pivot steps. Up to permutation of rows (not involving the last row) and permutation of columns (not involving the last column), there are at most $\frac{(m+n)!}{m!n!}$ tableaux. Both perturbation and Bland's rule prevent returning to a previous tableau up to the above permutations. So, if we use one of this methods, the total number of pivot steps (in both phases together) is less than $\frac{(m+n)!}{m!n!.}$

A better bound for the number of corners in the feasible region and hence for the number of pivot steps in Phase 2 can be obtained. E.g., when $n = 2$, row feasible region is given by $m+2$ linear constraints in the terms of 2 variables on the top margin, so it has at most $m+2$ corners hence Phase 2 terminates in at most $m+1$ pivot steps,

T2. *The dimension of feasible regions*

For any mathematical program, the dimension of the feasible region can be defined by the maximal number of linearly independent vectors of the form $x - y$ where x and y are feasible solutions. (For infeasible programs, the dimension is -1.) It does not exceed the number of unknowns. When the feasible region is given by linear equations, this dimension agrees with that given on p. 57, line 15 of the textbook. When the feasible region is convex, this dimension agrees with other dimesions defined in topology.

For the linear program given by a standard row tableau with matrix $\begin{matrix} A & b \\ c & d \end{matrix}$ of size $(m+1) \times (n+1)$, the dimension does not exceed n. When $b > 0$ or, more generally, the last nonzero entry in every of the first m rows is positive, the dimension is exactly n.

Note that the feasible region for a linear program in standard or canonical form does not contain straight lines.

T3. *Hypercubes*

Consider the standard tableau $\begin{bmatrix} A & b \\ c & d \end{bmatrix}$ where $A = -1_n$ and all entries of b are ones. In terms of n variables on top, the feasible region is an n-cube. It is given by $2n$ linear constraints (every variable on top is between 0 and 1), and it has 2^n corners. Its dimension is n. Every n-cibe is similar to this one. The 2-cubes are the squares. In fact all tableaus in this program has the same A and b, and there are 2^n of them. When n = 100, this number is $2^{100} = 1024^{10} =$ 1267650600228229401496703205376 which is smaller than the bound $\frac{(n+m)!}{m!n!} = \frac{200!}{100!^2} =$ 90548514656103281165404177077484163874504589675413336841320 for the number of corners in the feasible region for a linear program given by 101 by 101 standard tableau.

It was shown that under the most common ways to specify the choice of pivot entry in Phase 2, for some $(n+1) \times (n+1)$ feasible tableaux with 2^n corners in the feasible region, it takes exactly $2^n - 1$ pivot steps to reach an optimal tableau.

T4. *Simplex*

An n-dimensional simplex is given by $n+1$ linear constraints and has $n+1$ corners (vertices). Its dimension is n. When n = 1, it is an interval (the mixtures of two distinct points). When $n = 2$, it is a triangle. When $n = 3$, iy is a tetrahedron. A regular n-simplex consists of all $n+1$-columns $x \geq 0$ with the sum of entries equal to 1. In other words, its corners are the columns of the identity matrix 1_{n+1}. Any regular n-simplex is similar to that one. It can be given by $n+1$ linear constraints for n variables but it cannot be done with rational coefficients (or with rational corners) when $n \geq 2$. In Chapter 7, we will see the regular simplexes as the mixed strategies for the row player in matrix games.

T5. *When the feasible region is bounded*?

A subset in an Euclidean space is *bounded* if the is an upper bound for the distance between its points. This is equivalent to any of the following:

(a) any linear form on X is bounded from above,

(b) X is contained in a simplex;

(c) X is contained in a hypercube.

Given an optimal tableau, the feasible region is unbounded if an only if there is a non-negative nonzero column which is not the last column.

The feasible region for a linear program is bounded if and only if every feasible solution is a mixture of basic feasible solutions.

T6. *Polytopes or polyhedra*?

The feasible regions for linear programs are called convex polytopes, polyhedral sets, or polyhedra. In some publications all polytopes are required to be bounded. In other publications all polyhedra are bounded and 3-dimensional (e.g., tetrahedron, octahedron, dodecahedron, icosahedron).

1. The diamond can be given by four linear constraints $\pm x \pm y \leq 1$.

2. Any convex combination of convex combinations is a convex combination.

3. We have to prove that if $u = [x_1, y_1], v = [x_2, y_2]$ are feasible, i.e.,

$$x_1^4 + y_1^4 \leq 1 \text{ and } x_2^4 + y_2^4 \leq 1,$$

then the point $au + (1-a)v$ is also feasible for $0 \leq a \leq 1$, i.e.,

$$(ax_1 + (1-a)x_2)^4 + (ay_1 + (1-a)y_2)^4 \leq 1.$$

Clearly, it suffices to do this in the case when $x_1, x_2, y_1, y_2 \geq 0$. In other words, it suffices to prove that the region

$$x^4 + y^4 \leq 1, x \geq 0, y \geq 0$$

is convex. The function $y = (1 - x^4)^{1/4}$ is smooth on the interval $0 < x < 1$, so it suffices to show that its slope decreases. At the point $[x, y] = [x, (1 - x^4)^{1/4}]$ the slope is $-x^3/y^3$ so it decreases.

Similarly, we can prove that the region $|x|^p + |y|^p \leq 1$ is convex for any $p \geq 1$. In the case $p = 1$ the slope is -1, a constant function.

5. The points $[x, y] = [3, 1], [3, -1]$ belong to the circle but the halfsum $[3, 0]$ (the center of the circle) does not.

7. Both $x = 1$ and $x = -1$ belong to the feasible region, but $0 = x/2 + y/2$ does not.

8. The tangent to the disc at the point $[x, y] = [2t/(1+t^2), (1-t^2)/(1+t^2)]$ is $2tx + (1-t^2)y = 1 + t^2$. The family of linear constraints $2tx + (1-t^2)y \leq 1 + t^2$, where t ranges over all rational numbers, gives the disc.

9. A set S is called closed if it contains the limit points of all sequences in S. Any system of linear constraints gives a closed set, but the interval $0 < x < 1$ is not closed. Its complement is closed.

10. The rows of the identity matrix 1_6. If the vectors are written as columns, take the columns of 1_6.

11. One.

13. Since x_i are affine, (a) $\Rightarrow$ (b). It is also clear that (b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) $\Rightarrow$ (a). So it remains to prove that (c) $\Rightarrow$ (d). The last implication follows from the well-known inequality

$$(|x_1| + \cdots + |x_n|)/n \leq ((x_1^2 + \cdots + x_n^2)/n)^{1/2}.$$

14. Let x be in S is not a vertex. We find distinct y, z in S such that $x = (y+z)/2$. The linear constraints giving S restricted on the

line $ay + (1-a)z$ give linear constraints on a. The interval $0 \leq a \leq 1$ is a part of the feasible set. Any of the constraints is either tight on the whole interval, or is tight only at am end point. So the tight constraints are the same for all points $ay + (1-a)z$ with $0 < a < 1$.

The "only if" part proven, consider now the "if" part. Here is a counter example with an infinite system of constraints: The linear constraints are $x \geq c$ where c runs over all negative numbers. The feasible set S is the ray $x \geq 0$. No defining constraint is tight for any feasible x but $x = 0$ is the only vertex.

So we assume the S is given by a *finite* system of linear constraints. Let x, y be in S and x is a vertex and y has the same tight constraints. If $y \neq y$ then the same constraints are tight for every point on the line $ax + (1-a)y$. For a number $a_0 > 1$ sufficiently closed to 1, all other constraints are also satisfied (here we use the finiteness). We pick such a number $a_0 < 2$. Then

$$x = ((a_0 x + (1-a_0)y) + ((2-a_0)x + (a_0-1)y))/2$$

is not a vertex because $a_0 x + (1-a_0)y$ and $(2-a_0)x + (a_0-1)y$ are distinct points in S.

15. Suppose that x is optimal, y, z are in the convex set S, and $x = (y+z)/2$. We have to prove that $y = z$.

Since f is affine, $f(x) = (f(y) + f(z))/2$. Since x is optimal, so are y and z. By uniqueness of optimal solution, $y = x = z$.

17. Our set S is a subset of R^n. Let x be a point of S such that it is the limit of a sequence $y^{(1)}, y^{(2)}, \ldots.$ of points outside S (in other words, x belongs to the boundary of S). For example, x could be a vertex of S.

For each t, we find the point z in the closure of S closest to $y^{(t)}$. (We use the Euclidean distance $((y-z)\cdot(y-z))^{1/2}$ between points y, z in R^n.) We consider the linear constraint

$$(y^{(t)} - z^{(t)}) \cdot X \leq (y^{(t)} - z^{(t)})(y^{(t)} + y^{(t)})/2.$$

All points in S satisfy this constraint, while the point $y^{(t)}$ does not. Now we scale this constraint so it takes the form $c^{(t)} X \leq b^{(t)}$ with $c^{(t)} \cdot c^{(t)} = 1$. Then we take a limit constraint $c \cdot X \leq b$. Then all points in S satisfy the latter constraint and $c \cdot x = b$. Thus, x is maximizer of the linear form $f = c \cdot X \neq 0$ over S.

When the set S is closed, and x is a vertex, we can arrange x to be an unique maximizer.

18. Suppose that x is not a vertex in S'. Then $x = (y+z)/2$ with distinct $y, z.$ in S'. Since $y, z.$ are in S, x is not a vertex in S.

Chapter 5. Duality

§13. Dual Problems

T!. *About the theorem on 4 alternatives*
See
http://personal.psu.edu/lxv1/4alt.html

T2 *Redundant rows and columns in standard tableaux*

Consider a standard tableau with matrix $\begin{matrix} A & b \\ c & d \end{matrix}$. If a row of the matrix $[A, b]$ is ≥ 0, then the row is redundant for row program. Deleting this row does not change the feasible and optimality regions for the row problem. Also the optimal value (including symbols ∞ and $-\infty$) for the column problem does not change.

A more suttle redundancy happens when a row of $[A, b]$ is ≤ 0, (almost bad row). If the last entry is negative, it is a bad row so the row problem is infeasible (and the column problem is either infeasible or unbounded) hence all other rows in are redundant for the row problem. Assume now that the last entry is 0. Then we can drop this row together with the columns of corresponding to its positive entries without changing the optimal values of the row and column problems.

T3. *All feasible solutions*

Any feasible tableau cam be improved by pivot steps (and dropping redundant rows and columns) so it would show more explicitly all feasible solutions. An improved tableau has the last nonzero entry in each row (except the last row) positive.

T4. *All optimal solutions*

Any optimal tableau cam be improved by pivot steps (and dropping redundant rows and columns) so it would show more explicitly all optimal solutions. Firstly, the variables on top corresponding to the columns with positive last entries must be 0 in every optimal solution. So we drop these columns. Now the optimal solutions are the feasible solution, so see T3 above.

T5. *The formula* $(cx^T + d) - (-y^T b + d) = vx^T + y^T u$
for the standard tableau (13.4)holds without the sign constraints $x \geq 0, y \geq 0, u \geq 0, v \geq 0$. Without these constrains the numbers $cx^T + d$ and $-y^T b + d$ need not to be feasible values and the number $vx^T + y^T u$ could be negative.

T6. *Reduction to Phase* 2

In Remark 15.2, using duality,, we saw that finding an optimal solution for a linear program can be reduced to finding a feasible solution for another linear program (reduction to Phase 1). Now we show that Phase 1 can be reduced to Phase 2. Suppose we want to find a feaible solution for $Ax \le b, x \ge 0$, where A is an m by n matrix. Consider the linear program $Ax \le b, t \le 1, x \ge 0, t \ge 0, -t \to \min$ with a new unknown t. This LP has $x = 0, t = 0$ as a basic feasible solution, and it is bounded. If tits optimal value is 0, then the original problem has no feasible solutions. Otherwise, the optimal solutions for LP have $t = 1$ and their x-parts are the feasible solutions for $Ax \le b, x \ge 0$,

1. Let $f = 5x - 6y + 2z$ be the objective function. Here is a standard column tableau:

$$\begin{array}{c} -x \\ -y \\ -z \\ 1 \end{array} \begin{bmatrix} 0 & -1 & 5 \\ 1 & -1 & -6 \\ 0 & 0 & 2 \\ 7 & -3 & 0 \end{bmatrix}$$
$$\begin{array}{cccc} \shortparallel & \shortparallel & \downarrow & \\ v_1 & v_2 & -f & \to \max \end{array}$$

3. $$\begin{array}{cccc} x & y & z & 1 \end{array}$$
$$\begin{bmatrix} 0 & -1 & 0 & 7 \\ 1 & 1 & 0 & -3 \\ 5 & -6 & 2 & 0 \end{bmatrix} \begin{array}{l} = v_1 \\ = v_2 \\ = f \quad \to \min \end{array}$$

The matrix in Exercise 2 is so big that the transposed matrix may not fit on the page. So we reduce it as follows. The fifth constraint follows from the fourth constraint because $b \ge 0$ so we drop the redundant constraint.

Given any feasible solution, we can replace g, h by $0, g + h$ and obtain a feasible solution with the same value for the objective function f. So setting $g = 0$ we do not change the optimal value.

$$\begin{array}{cccccccccc} a & b & c & d & e & f & h & i & j & 1 \end{array}$$
$$\begin{bmatrix} 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 10 & -100 \\ 25 & 25 & 25 & 0 & 6 & 25 & 25 & 0 & 10 & -100 \\ 1 & 25 & 25 & 25 & 0 & 30 & 25 & 25 & 25 & -100 \\ 1 & 0 & 25 & 25 & 30 & 25 & 25 & 25 & 25 & -100 \\ 3 & 2 & 1 & 2 & 4 & 2 & 2 & 3 & 3 & -70 \\ .39 & .11 & .18 & .21 & .35 & .44 & .25 & .23 & .24 & 0 \end{bmatrix} \begin{array}{l} = v_1 \\ = v_2 \\ = v_3 \\ = v_4 \\ = v_6 \\ \to \min \end{array}$$

5. Let $cx+d, cy+d$ be two feasible values, where x, y are two feasible solutions. We have to prove that

$\alpha(cx+d)+(1-\alpha)(cy+d)$

is a feasible value for any α such that $0\alpha \le 1$. But

$\alpha(cx+d)+(1-\alpha)(cy+d) = c(\alpha x+(1-\alpha)y)+d,$

where $\alpha x+(1-\alpha)y$ is a feasible solution because the feasible region is convex.

7. The first equation does not hold, so this is not a solution.

8. First we check that this $X=[x_i]$ is a feasible solution (i.e., satisfies all constraints) with $z=2$. We introduce the dual variables y_i corresponding to x_i, write the dual problem as the column problem, and set $y_i=0$ whenever $x_i \neq 0$ (assuming that X is optimal, cf. Problem 13.10 and its solution).

$$\begin{array}{r} -y_6 \\ -y_8 \\ 1 \\ \\ \end{array}\begin{array}{c}\left[\begin{array}{cccccc} 7 & -2 & -6 & 6 & -1 & 15 \\ -1 & 0 & 1 & 0 & -1 & -2 \\ 4 & 0 & -3 & 5 & 3 & 4 \end{array}\right] \\ \begin{array}{cccccc} =0 & =y_2 & =0 & =y_4 & =0 & =w \end{array} \end{array} \to \max .$$

We have a system of three linear equations for y_6, y_8 , and the system has no solutions. So X is not optimal.

9. Proceeding as in the solutions of Problem 13.10 and Exercise 8, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1[\begin{bmatrix} 7 & -2 & -6 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ 4 & 0 & -3 \end{bmatrix} = 0.$$

The system has the unique solution $y_6, = 1, y_7 = 2, y_8 = 1$. Moreover, this solution is feasible (the basic y_i are nonnegative). Since we have feasible solutions for the primal and dual problems and $x_i y_i = 0$ for all i, both solutions are optimal.

10. Proceeding as in the solutions of Problem 13.10 and Exercises 8,9, we obtain the following system of linear equations:

$$[-y_6, -y_7, -y_8, 1[\begin{bmatrix} -2 & -6 & 6 & -1 \\ 1 & 1 & -2 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 5 & 3 \end{bmatrix} = 0.$$

The system has no solutions, so the answer is: This is not optimal.

§14. Sensitivity Analysis and Parametric Programming

T1. *How the optimal value dependsof the constant in the objective function*

Replacing the number d in the initial standard tableau with the matrix $\begin{bmatrix} A & b \\ c & d \end{bmatrix}$ by d' results in replacing the optimal value $\bar{d}$ by $\bar{d}+d'-d$. So $\bar{d}$ is an affine function of the constant d in the objective function.

T2. *How the optimal value depends on the last column and row of the initial standard tableau*

Suppose we make a few pivot steps starting with a standard tableau with the matrix $\begin{bmatrix} A & b \\ c & d \end{bmatrix}$, and finishing with a new standard tableau with the matrix $\begin{bmatrix} \bar{A} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$. Then $\bar{b} = Bb$ and $\bar{c} = cC$, where the matrices B and C depend only on A and our choices of pivot entries. Furthermore, $\bar{d} = d - \tilde{c}\tilde{b}$, where $\tilde{b}$ is the part of b corresponding to the variables that moved to the top margin and $\tilde{c}$ is the part of $\bar{c}$ corresponding to these variables.

Suppose now that the new matrix is optimal. Then it stays optimal in the region $\bar{b} = Bb \geq 0$. In particular, when $\bar{b} > 0$, the entries of the row $\tilde{c}$, i.e., entries of the optimal solution for the column linear program, are the slopes of the optimal value $\bar{d}$ as function of the entries of b.

If $\bar{b}$ has zero entries, we may have more than one basic optimal solution and more than one slope.

The whole region of b for which we have the optimal value is given by a finite system of linear constraints on b. and it is covered by a finite set of tableaux $\begin{bmatrix} \bar{A} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$ each of them is optimal in the region $Bb \geq 0$.

1. The tableau is not standard, so we treat the row and column programs separately. We pivot the row program on -1 in the b-column:

$$\begin{array}{c} \begin{array}{cccc} a & b & c & 1 \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & -1 & -2 \\ 2 & -1^* & 0 & -3 \\ 0 & 2 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ = d \\ = c \\ = w \quad \to \min \end{array} \qquad \mapsto$$

$$\begin{array}{cccccl} a & c & c & 1 & & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 2 & -1 & 0 & -3 \\ 4 & -2 & 1 & -6 \end{bmatrix} \begin{array}{l} = d \\ = b \\ = w \quad \to \min . \end{array}$$

Then we combine the two c-columns:

$$\begin{array}{c} \begin{array}{ccc} a & c & 1 \end{array} \\ \begin{bmatrix} 1 & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{bmatrix} \end{array} \begin{array}{l} = d \\ = b \\ = w \quad \to \min . \end{array}$$

This tableau is standard. We use the simplex method:

$$\begin{array}{c} \begin{array}{ccc} a & c & 1 \end{array} \\ \begin{bmatrix} 1^* & -1 & -2 \\ 2 & -1 & -3 \\ 4 & -1 & -6 \end{bmatrix} \end{array} \begin{array}{l} = d \\ = b \\ = w \quad \to \min \end{array} \quad \mapsto$$

$$\begin{array}{c} \begin{array}{ccc} d & c & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \end{array} \begin{array}{l} = a \\ = b \\ = w \quad \to \min . \end{array}$$

This is an optimal tableau, so $\min = 2$ at $a = 2, b = 1, c = d = 0$.

Now we rewrite the column program in a standard column tableau:

$$\begin{array}{c} -g \\ -h \\ 1 \\ \end{array} \begin{array}{c} \begin{bmatrix} -1 & 0 & 1 & 2 \\ -2 & 1 & 0 & 3 \\ 0 & -2 & -1 & 0 \end{bmatrix} \\ \begin{array}{cccc} = i & = j & = k & = -u \end{array} \quad \to \quad \max . \end{array}$$

The k-column is bad, so this program is infeasible.

3. $\min = 0$ at $d = e = 0, a \geq 0$ arbitrary

§15. More on Duality

1. No, it is not redundant.
2. Yes, it is 2· (first equation) + (second equation).
3. No, it is not redundant.
4. No, it is not redundant.
5. Yes, it is Adding the first two equations, we obtain

$$6x + 8y + 10z = 12$$

which implies the last constraint because $12 \geq 0$.

7. Yes, it is.

11. Let y_i be the dual variable corresponding to x_i. The first 7 columns of the tableau give 7 linear constraints for y_8, y_9. Two additional constraints are $y_8, y_9 \geq 0$. We can plot the feasible region (given by these 9 constraints) in the (y_8, y_9)-plane. (The constraints corresponding to x_5, x_6, x_7 are redundant.) The answer is max $= 2.5$ at $y_8 = 0, y_9 = 1.25$. By complementary slackness, for any optimal solution $[x_i]$ of the primal problem, we have $x_i = 0$ for $i \neq 1, 8$. For such a solution, we have $3x_1 - 1 = x_8 \geq 0, 4x_1 - 2 = x_9 = 0$. Thus,

min $= 2.5$ at $x_1 = 0.5$, all other $x_i = 0$.

13. Let y_i be the dual variable corresponding to x_i, and let u, v be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for u, v. Two additional constraints are $u, v \geq 0$. We can plot the feasible region (given by these 11 constraints) in the (u, v)-plane. The answer is max $= 75/34$ at $u = 15/34, y_{10} = 15/17$. By complementary slackness, for any optimal solution $[x_i]$ of the primal problem, we have $x_i = 0$ for $i \neq 4, 6$. For such a solution, we have $6x_4 + 8x_6 - 1 = 0, 14x_4 + 13x_6 - 2 = 0$. Thus,

min $= 75/34$ at $x_4 = 3/34, x_6 = 1/17$ all other $x_i = 0$.

14. Let y_i be the dual variable corresponding to x_i, and let u, v be the nonbasic dual variables (corresponding to the first two rows of the tableau). The first 9 columns of the tableau give 9 linear constraints for u, v. Two additional constraints are $u, v \geq 0$. The constraint corresponding to the x_8 column reads $-5u - 8v + 0 \geq 0$, hence $u = v = 0$. On the other hand, then the constraint corresponding to the x_1 column reads $-1 \geq 0$. So the column problem is infeasible.

On the other hand, it is easy to find a feasible solution for the row program, for example, $x_9 = 1$ and $x_i = 0$ for all other i. By the theorem on four alternatives, the row problem is unbounded.

Chapter 6. Transportation Problems

§16. Phase 1

T1, *One row or column*

In any closed transportation problem with only one retail store or only one warehouse, there is exactly one feasible solution. It is optimal. To get a basic feasible solution, we select all positions.

1.

20	10	5		35
		5	15	20
20	10	10	15	

3. The total supply is 256, while the total demand is 260. So the problem is infeasible.

5. The balance condition $50 = 50$ holds. Each time, we pick a position with the minimal cost: $x_{21} = 2, x_{24} = 13, x_{33} = 4$ at zero cost, $x_{14} = 2, x_{17} = 3, x_{49} = 1, x_{42} = 7, x_{32} = 1, x_{38} = 4, x_{36} = 2$ at unit cost 1, and $x_{15} = 10, x_{35} = 1$ at unit cost 2. The total number of selected positions is 12, which equals $4+9-1$. Total cost is $0 \cdot 19 + 1 \cdot 20 + 2 \cdot 11 = 42$.

6. The balance condition $50 = 50$ holds. Each time, we pick a position with the minimal cost: $x_{21} = 5, x_{33} = 12$ at zero cost, $x_{14} = 5, x_{17} = 3, x_{49} = 1, x_{42} = 8, x_{41} = 7, x_{43} = 2$ at unit cost 1, $x_{15} = 1, x_{18} = 4, x_{46} = 0$ at unit cost 2, and $x_{16} = 2$ at unit cost 3. The total number of selected positions is 12, which equals $4 + 9 - 1$. Total cost is $0 \cdot 17 + 1 \cdot 26 + 2 \cdot 5 + 3 \cdot 2 = 42$.

7. The balance condition $130 = 130$ holds. Each time, we pick a position with the minimal cost: $x_{16} = 30$ at zero cost, $x_{12} = 10, x_{35} = 15$ at unit cost 30, $x_{21} = 25, x_{34} = 30$ at unit cost 35, $x_{13} = 10$, at unit cost 40, $x_{33} = 5$, at unit cost 95, and $x_{23} = 5$, at unit cost 100. The total number of selected positions is 8, which equals $3+6-1$. Total cost is $0 \cdot 30 + 30 \cdot 25 + 35 \cdot 55 + 40 \cdot 10 + 95 \cdot 5 + 100 \cdot 5 = 4050$.

§17. Phase 2

T1. *What data are allowed?*

It is implicitly assumed that both supply and demand in any transportation problem are non-negative. Since $x_{i,1} + \cdots x_{i,n} \le a_i$ for every i, a negative supply a_i would make the problem infeasible.On the other hand, the negative $b_j s$ can be replaced by zeros without changing the feasible region or the optimality region.

So a transportation problem is feasible if and only if the total supply is $\geq$ total demand.

1.

	1	2	2	
0	1 175	2 25	3 (1)	200
0	1 (0)	2 100	2 200	300
	175	125	200	

This is an optimal table, and the corresponding solutions are optimal. The minimal cost for the transportation problem is $1 \cdot 175 + 2 \cdot 25 + 2 \cdot 100 + 2 \cdot 200 = 825$. The maximal profit for the dual problem is $1 \cdot 175 + 2 \cdot 125 + 2 \cdot 200 - 0 \cdot 200 - 0 \cdot 300 = 825$.

3. We start with the basic feasible solution found in the solution of Exercise 4, §16 and compute the corresponding dual basic solution:

	0	2	1	0	1	2	1	2	1	
0	1 (1)	2 (0)	3 (2)	1 (1)	2 (1)	3 (1)	1 3	2 7	3 (2)	10
0	0 2	3 (1)	2 (1)	0 5	1 1	2 (0)	1 (0)	2 (0)	1 12	20
1	2 (3)	1 3	0 4	1 (2)	2 (2)	1 (0)	2 (2)	1 5	1 (1)	12
0	1 (1)	1 (-1)	1 (0)	2 (2)	2 (1)	2 2	2 (1)	2 2	1 4	8
	2	3	4	5	1	2	3	14	16	

There is only one negative $w_{42} = -1$. We select this position and get the loop (4, 2), (3, 2), (3, 8), (4, 8). The maximal $\varepsilon = 2$, and we deselect the position (4, 8). The total cost decreases by 2. Here is the new basic feasible solution and the corresponding dual basis solution:

	1	2	1	1	2	3	1	2	2	
0	1 (0)	2 (0)	3 (2)	1 (0)	2 (0)	3 (0)	1 3	2 7	3 (1)	10
1	0 2	3 (2)	2 (2)	0 5	1 1	2 (0)	1 (1)	2 (1)	1 12	20
1	2 (2)	1 1	0 4	1 (1)	2 (1)	1 (-1)	2 (2)	1 7	1 (0)	12
1	1 (1)	1 2	1 (1)	2 (2)	2 (1)	2 2	2 (2)	2 (1)	1 4	8
	2	3	4	5	1	2	3	14	16	

Again we have a negative $w_{36} = -1$. The loop is (3, 6), (3, 2), (4, 2), (4,6). The total cost decreases by $-w_{36} \cdot \varepsilon = 1$. Here is the new basic feasible solution and the corresponding dual basis solution:

	0	1	1	0	1	2	1	2	1	
0	1 (1)	2 (1)	3 (2)	1 (1)	2 (1)	3 (1)	1 3	2 7	3 (2)	10
0	0 2	3 (2)	2 (1)	0 5	1 1	2 (0)	1 (0)	2 (0)	1 12	20
1	2 (3)	1 (1)	0 4	1 (2)	2 (2)	1 1	2 (2)	1 7	1 (1)	12
0	1 (1)	1 3	1 (0)	2 (2)	2 (1)	2 1	2 (1)	2 (0)	1 4	8
	2	3	4	5	1	2	3	14	16	

This table is optimal with total cost at min = 47.

T2. *Negative prices*

The prices $c_{i,j}$s in a closed (balanced) transportation problem need not to be non-negative. On the other hand, for an open (unbalanced) transportation problem, negative $c_{i,j}$s make the balancing trick (adding a dummy, a fictitious store with zero cost of shipping to it) invalid. A way which work is to make price $c_{i,0}$ of shipping from warehouse i to the dummy to be $\max(0, \min_j(c_{i,j}))$. After the closed problem is solved, we reroute the flow $x_{i,0}$ from i to 0 with $c_{i,0} < 0$ to j with $c_{i,j} = c_{i,0}$. This does not change the optimal value.

5. We start with the basic feasible solution found in the solution of Exercise 6, §16 and compute the corresponding dual basic solution:

	2	2	2	1	2	3	1	2	2	
0	1 (-1)	2 (0)	3 (1)	1 5	2 1	3 2	1 3	2 4	3 (1)	15
2	0 5	3 (3)	2 (2)	0 (1)	1 (1)	2 (1)	1 (2)	2 (2)	1 (1)	5
2	2 (2)	1 (1)	0 12	1 (2)	2 (2)	1 (0)	2 (3)	1 (1)	1 (1)	12
1	1 7	1 8	1 2	2 (2)	2 (1)	2 0	2 (2)	2 (1)	1 1	18
	12	8	14	5	1	2	3	4	1	

There is only one negative $w_{11} = -1$. The loop is (1,1), (1, 6), (4, 6), (4, 1). The decrease in the total cost is 2. Here is the new basic solution and the corresponding dual basic solution:

	1	1	1	1	2	2	1	2	1	
0	1 2	2 (1)	3 (2)	1 5	2 1	3 (1)	1 3	2 4	3 (2)	15
1	0 5	3 (3)	2 (2)	0 (0)	1 (0)	2 (1)	1 (1)	2 (1)	1 (1)	5
1	2 (2)	1 (1)	0 12	1 (1)	2 (1)	1 (0)	2 (2)	1 (0)	1 (1)	12
0	1 5	1 8	1 2	2 (1)	2 (0)	2 2	2 (1)	2 (0)	1 1	18
	12	8	14	5	1	2	3	4	1	

This table is optimal, and min = 40.

7. When $t \geq 25$, see the previous solution. When $t < 0$, the total supply is less than the total demand, so the program is infeasible. So it remains to consider the case $0 \leq t \leq 25$.

T3. *Bounded and feasible*

Any transportation problem is bounded, namely,

$\sum_{i,j} c_{i,j} x_{i,j} \geq \sum a_i \min_j c_{i,j}$.

It is feasible if and only if

total supply is $\geq$ total demand

in which case there is an optimal solution.

T4. *Tips for selecting positions inPhase* 1
See
https://bisor.wiwi.uni-kl.de/orwiki/Transportation_problem:_Construction_of_starting_solution_1

https://bisor.wiwi.uni-kl.de/orwiki/Transportation_problem:_Construction_of_starting_solution_2

http://www.universalteacherpublications.com/univ/ebooks/or/Ch5/vogel.htm
about different ways to find a basic feasible solution.

8. In the optimal tableau of Table 17.22, we replace c_{23} by t and w_{23} by $t - 40$:

	35	30	40	35	30	0	
0	55 (20)	30 10	40 20	50 (15)	40 (10)	0 20	50
0	35 25	30 (0)	t $(t-40)$	45 (10)	60 (30)	0 5	30
0	40 (5)	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

When $t \geq 40$, this table is optimal, with min = 3475 (independent of t).

Assume now that $t < 40$. We select the position (2, 3) and get the loop (2, 3), (1,3), (1,6), (2,6). The corresponding $\varepsilon = 5$. The decrease in the total cost is $5(40 - t)$. Here is the new table:

	$75-t$	30	40	35	30	0	
0	55 $(t-20)$	30 10	40 15	50 (15)	40 (10)	0 25	50
40 $-t$	35 25	30 $(40-t)$	t 5	45 $(50-t)$	60 $(70-t)$	0 $(40-t)$	30
0	40 $(t-35)$	60 (30)	95 (55)	35 30	30 15	0 5	50
	25	10	20	30	15	30	

This table is optimal when $35 \leq t \leq 40$.

Assume now that $t < 35$. We select the position (3, 1) and get the loop (3, 1), (3, 6), (1,6), (1, 3), (2,3), (2,1). The corresponding $\varepsilon = 5$. The decrease in the total cost is $5(35 - t)$. Here is the new table:

	$75 - t$	30	40	$70 - t$	$65 - t$	0
0	55 $(t - 20)$	30 10	40 10	50 $(t - 20)$	40 $(t - 25)$	0 30
40 $-t$	35 20	30 $(40 - t)$	t 10	45 (15)	60 (35)	0 $(40 - t)$
35 $-t$	40 5	60 $(65 - t)$	95 $(90 - t)$	35 30	30 15	0 $(35 - t)$
	25	10	20	30	15	30

This table is optimal when $25 \leq t \leq 35$.

Assume now that $t < 25$. We select the position (1, 5) and obtain the loop (1, 5), (3, 5), (3, 1), (2, 1), (2, 3), (1, 3). The corresponding $\varepsilon = 10$. The decrease in the total cost is $10(25 - t)$. Here is the new table:

	50	30	$t + 15$	45	40	0	
0	55 (5)	30 10	40 $(25 - t)$	50 (5)	40 10	0 30	50
15	35 10	30 (15)	t 20	45 (15)	60 (35)	0 $40 - t$	30
10	40 15	60 (40)	95 $(90 - t)$	35 30	30 5	0 (10)	50
	25	10	20	30	15	30	

This table is optimal when $t \leq 25$.

Thus, we solve the program for all t. The minimal cost is

$$\begin{cases} 2900 + 20t & \text{for } t \leq 25 \\ 3400 + 5(t - 25) & \text{for } t \geq 25 \\ 3475 & \text{for } t \geq 40. \end{cases}$$

T1. *Basic solutions are integral*

If every supply a_i and every demand b_j in a feasible transportation problem is an integer, then the simplex method find an optimal solution in integers.

§18. Job Assignment Problem

T1. *Why Hungarian*?

W. Kuhn called his method (published in 1955 and 1956) of solving the job assignment problems "Hungarian Method" because he used ideas of two Hungarian mathematicians, Dénoes König and Jenö Egenváry. James Munkres examined the running time of the method in 1957 and observed that it is strongly polynomial. Edmonds and Karp, and independently Tomizawa improved running time. Ford and Fulkerson extended the method to general transportation problems. In 2008, it was discovered that Carl Gustav Jacobi knew the method. His paper appeared posthumously in 1890 in Latin.

For more details, see

http://en.wikipedia.org/wiki/Hungarian_method

and

Kuhn, Harold W. A tale of three eras: the discovery and rediscovery of the Hungarian Method. European J. Oper. Res. 219 (2012), no. 3, 641651. 90-03 (01A55 01A60 90C27). MR2898945.

1. min = 7 at $x_{14} = x_{25} = x_{32} = x_{43} = x_{51} = 1$, all other $x_{ij} = 0$.

3. min = 7 at $x_{12} = x_{25} = x_{34} = x_{43} = x_{51} = x_{67} = x_{76} = 1$, all other $x_{ij} = 0$.

4. We subtract: 1 from the rows $2, 3, 4, 5, 8$; 2 from the row 7; 1 from the column 8:

$$\begin{array}{|ccccccccc|}
\hline
0 & 2 & 2 & 4 & 0 & 1 & 5 & 0 & 4 \\
1 & 2 & 0 & 2 & 3 & 2 & 1 & 2 & 0 \\
1 & 0 & 3 & 2 & 3 & 2 & 0 & 0 & 3 \\
3 & 1 & 3 & 1 & 1 & 1 & 0 & 3 & 1 \\
1 & 2 & 4 & 0 & 1 & 3 & 1 & 2 & 3 \\
5 & 2 & 2 & 4 & 0 & 1 & 1 & 0 & 4 \\
0 & 1 & 2 & 1 & 2 & 1 & 0 & 1 & 3 \\
4 & 1 & 3 & 2 & 1 & 0 & 1 & 2 & 0 \\
1 & 2 & 1 & 2 & 0 & 1 & 2 & 3 & 5 \\
\hline
\end{array}.$$

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Hungarian method (Remark 17.25). Let us try to place the flow at positions with zero cost. In each of the following six lines there is only one zero: r4 (row 4), r5, r9, c2 (column 2), c3, c6 (we pass c4 because the conflict with r5: the position (5,4) is already selected in r5). We select the positions with these zeros and add the six lines c7, c4, c5, r3, r2, r8 to our list L of covering lines. The remaining matrix is

$$\begin{array}{cccc} & c1 & c8 & c9 \\ r1 & 0 & 0 & 4 \\ r6 & 5 & 0 & 4 \\ r3 & 0 & 1 & 3 \end{array}.$$

We cannot place all flow at positions with zero cost, but we can cover all zeros by $2 < 3$ lines, namely, c1 and c8. The complete list L consists of 8 lines c7, c4, c5, r3, r2, r8, c1, c8. The least uncovered number is $m = 1$. We subtract 1 from all uncovered entries and add 1 to all twice-covered entries:

$$\boxed{\begin{array}{ccccccccc} 0 & 1 & 1 & 4 & 0 & 0 & 5 & 0 & 3 \\ 2 & 2 & 0 & 3 & 4 & 2 & 2 & 3 & 0 \\ 2 & 0 & 3 & 3 & 4 & 2 & 1 & 1 & 3 \\ 3 & 0 & 2 & 1 & 1 & 0 & 0 & 3 & 0 \\ 1 & 1 & 3 & 0 & 1 & 2 & 1 & 2 & 2 \\ 5 & 1 & 1 & 4 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 \\ 5 & 1 & 3 & 3 & 2 & 0 & 2 & 3 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 & 2 & 3 & 4 \end{array}}.$$

Now we can place the flow at the positions with zero costs:

$$\boxed{\begin{array}{ccccccccc} & & & & & * & & & \\ & & * & & & & & & \\ & * & & & & & & & \\ & & & & & & * & & \\ & & & * & & & & & \\ & & & & & & & * & \\ * & & & & & & & & \\ & & & & & & & & * \\ & & & & * & & & & \end{array}}.$$

For this program, min $= 0$. However we changed the objective function (without changing the optimal solutions). For the original problem, min $= 1+1+1+1+1+1+2+1+0 = 9$.

5. max $= 14$ at $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$, all other $x_{ij} = 0$.

6. First we convert the maximization problem to a minimization problem by subtracting each entry from the maximal entry in its row:

$$\begin{array}{|cccccc|}\hline 0 & 2 & 2 & 2 & 3 & 2 \\ 2 & 1 & 3 & 2 & 0 & 0 \\ 2 & 4 & 3 & 3 & 3 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 0 & 3 \\ 1 & 0 & 1 & 0 & 2 & 1 \\ \hline \end{array}\,.$$

Since all entries are nonnegative, and there is at least one zero in each row and in each column, we are ready to apply the Hungarian method (Remark 17.25). The following five lines cover all zeros: rows 1,4,6 and columns 5,6. Since $t = 5 < n = 6$, we cannot place all flow at positions with zero cost. The least uncovered number is $m = 1$. We subtract 1 from all uncovered entries and add 1 to all twice-covered entries:

$$\begin{array}{|cccccc|}\hline 0 & 2 & 2 & 2 & 4 & 3 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 3 & 2 & 2 & 3 & 0 \\ 2 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 3 & 2 \\ \hline \end{array}\,.$$

Now we can place all flow at positions with zero cost:

$$\begin{array}{|cccccc|}\hline * & & & & & \\ & * & & & & \\ & & & & & * \\ & & * & & & \\ & & & & * & \\ & & & * & & \\ \hline \end{array}\,.$$

For this program, min = 0. However we changed the objective function (without changing the optimal solutions). For the original problem, max $= 4 + 3 + 4 + 4 + 4 + 2 = 21$.

Note that the sum of maximal entries in rows is 22, and we cannot get this much because the conflict in the last column. This proves that the solution is optimal.

7. max $= 30$ at $x_{15} = x_{26} = x_{33} = x_{41} = x_{54} = x_{62} = x_{77} = 1$, all other $x_{ij} = 0$.

Chapter 7. Matrix Games

§19. What are Matrix Games?

T1. *Optimal strategies*

Lets call his (her) strategy optimal if it maximizes his (reps., her) worst-case payoff. At any equilibrium, α, β both strategies involved are optimal. Indeed, let us compare the worst case-payoffs a and c for his strategy α, γ.

$$\begin{array}{ccc} \alpha & a^{*\blacksquare} & ? \\ \gamma & b & c^{\blacksquare} \end{array}.$$

We have $a \geq b \geq c$ hence $a \geq c$.

Conversely suppose that we have optimal strategies α, β for him and her. We will see in the next section that they exit and the optimal value a is the same for both optimization problems. Then we have:

$$\begin{array}{ccc} & ? & \beta \\ ? & ? & a^{*} \\ \alpha & a^{\blacksquare} & b \end{array}.$$

Since $a \leq b \leq a$, we have $b = a$. Thus, (α, β) is an equilibrium.

T2. *Games which are not matrix games*

A matrix game is a special case of a game in normal (strategic) form which is given by a set of players, a set of strategies for each player, and the payoff functions which give a number for each player when every player chooses a strategy. The concept of equilibrium makes sense for any game in normal form. However the equilibrium does not solve a game unless it is a two-player constant-sum game. We may have equilibria with different payoffs so we do not have definition for value or values of games. Even when the equilibrium is unique, it does not solve the game.

More restrictive concepts of strict equilibrium and strict domination can be introduced (for games in normal form). For strict equilibrium, we replace "no player can gain" in definition of equilibrium by "every player looses". For strict domination, we replace "better or equal" in definition of domination by "better".

A strategy which is strictly dominated cannot be a part of any equilibrium. Therefore elimination by strict domination does not change the set of equilibria.

A strategy which is dominated by a different strategy cannot be a part of any strict equilibrium.. Therefore elimination by domination does not change the set of strict equilibria. On the other hand, elimination by domination, while do not creating new equilibria, may decrease the set of equilibria. It may eliminate all equilibria in pure strategies as the following example shows.

	$c1$	$c2$
$r1$	$(1^*, -1)$	$(-1, 1^*)$
$r2$	$(-1, 1^*)$	$(1^*, -1)$
$r3$	$(-1, 1)$	$(1^*, 1^*)$

This is a two-player general-sum game. The only equilibrium in pure strategies is (r3, c2). The strategy r3 is (weakly) dominated by r2. If we eliminate r3, we have no equilibria in pure strategies.

Here are some popular videos:
https://www.youtube.com/watch?v=p3Uos2fzIJ0
https://www.youtube.com/watch?v=S0qjK3TWZE8
https://www.youtube.com/watch?v=t9Lo2fgxWHw
https://www.youtube.com/watch?v=uX4CkBlXoxo

1. max min $= -1$. min max $= 0$. There are no saddle points.
$[1/3, 2/3, 0]^T$ gives at least $-2/3$ for the row player.
$[1/2, 0, 0, 0, 1/2]$ gives at least $1/2$ for the column player.
So $-2/3 \le$ the value of the game $\le -1/2$.

3. max min $= -1$. min max $= 2$. There are no saddle points.
(second row + 2·third row)/3 $\ge -2/3$.
(third column + sixth column)/2 ≤ 1.
So $-2/3 \le$ the value of the game ≤ 1.

5. We compute the max in each column (marked by *) and min in each row (marked by $^{\blacksquare}$).

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} 4 & 2^{\blacksquare} & 3 & 5 & 4 & 3^{\blacksquare} & 7 & 6 & 3 \end{array} \\
\begin{array}{c} -4 \\ -2 \\ -4 \\ 0^* \\ -9 \end{array} &
\begin{bmatrix}
4^* & -4^{\blacksquare} & 3^* & 0 & 0 & 0 & -1 & 1 & -2 \\
-1 & 0 & 2 & 1 & -2^{\blacksquare} & -2^{\blacksquare} & 1 & 0 & -2^{\blacksquare} \\
-4^{\blacksquare} & 0 & -2 & -2 & 1 & -1 & 1 & 6^* & 2 \\
1 & 2^* & 2 & 5^* & 3 & 3^* & 7^* & 2 & 0^{\blacksquare} \\
-4 & -9^{\blacksquare} & -8 & 0 & 4^* & 2 & 2 & 0 & 3^*
\end{bmatrix}
\end{array}$$

Thus, max min $= 0$. min max $= 2$. There are no saddle points.

7. Optimal strategies are
$[0, 1/3, 0, 0, 7/15, 1/5]^T, [0, 2/3, 0, 0, 0, 0, 0, 0, 0, 1/3]$,
and the value of game is $-2/3$.

9. Optimal strategies are
$[55/137, 0, 0, 44/137, 0, 0, 38/137]^T$,
$[0, 0, 20/137, 0, 65/137, 0, 0, 0, 0, 0, 0, 0, 52/137]$,
and the value of game is $-54/137$.

11. We have seen that $a_{i,j} = a_{i',j} = a_{i,j'} = a_{i',j'}$ because $a_{i,j} \geq a_{i',j} \geq a_{i',j'} \geq a_{i,j'} \geq a_{i,j}$. Since $a_{i,j}$ and $a'_{i',j}$ are maximal in their columns j and j', so are $a_{i',j}$ and $a_{i,j'}$. Since $a_{i,j}$ and $a'_{i',j}$ are minimal in their columns i and i', so are $a_{i',j}$ and $a_{i,j'}$. Thus, (i, j') and (i', j) are saddle points.

§20. Matrix Games and Linear Programming

T1. *On Example* 20.1

The choice r1&c1 of the first pivot entry in Example 20.1 is not not consistent with our simplex method but it is consistent with our dual simplex method.. Choices consistent with the simplex method are r2&c1 and then r1&c2. They result in the same equilibrium.

1. The optimal strategy for the row player is $[2/3, 1/3, 0]^T$.
The optimal strategy for the column player is $[1/2, 1/2, 0]$.
The value of the game is 2.

3. The optimal strategy for the row player is $[0.2, 0, 0.8]^T$.
An optimal strategy for the column player is $[0, 0.5, 0.5, 0, 0, 0]$.
The value of the game is 1.

5. The optimal strategy for the row player is $[1/3, 2/3, 0]^T$.
The optimal strategy for the column player is $[2/3, 0, 0, 0, 1/3]$.
The value of the game is $-2/3$.

7. The optimal strategy for the row player: $[1/8, 0, 7/8, 0]^T$.
The optimal strategy for the column player: $[0, 1/4, 0, 0, 0, 3/4]$.
The value of the game is -0.25.

9. Optimal strategies are
$[0, 0, 0, 7/8, 1/8]^T, [3/8, 0, 0, 0, 0, 0, 0, 0, 5/8]$,
and the value of game is 3/8.

11. Note that the second constraint is redundant, because it follows from the first one. We solve the equation for x_6 and exclude this from our LP. We obtain an equivalent LP with all $x_i \geq 0$:
$10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5 - 9 \to \min$,
$3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4$,
$(x_6 + 3 =)\ 3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 3$.
Again, the second constraint is redundant.

Now we take advatage of the fact that in this problem all coefficients in the objective function and all right-hand parts of constraints are positive. We set $v = 1/(10x_1 + 5x_2 + 4x_3 + 7x_4 + 4x_5) > 0$ on the feasible region and
$p_1 = 10x_1 v, p_2 = 5x_2 v, p_3 = 4x_3 v, p_4 = 7x_4 v, p_5 = 4x_5 v$.

All $p_i \geq 0$ and $p_1 + p_2 + p_3 + p_4 + p_5 = 1$. The minimization of $1/v - 9$ is equivalent to the maximization of v.

The constraint $3x_1 + x_2 + x_3 + 2x_4 + x_5 \geq 4$ take the form

$$(3p_1/10 + p_2/5 + p_3/4 + 2p_4/7 + p_5/4)/4 \geq v.$$

This is the row player program for the matrix game with the payoff matrix $\begin{bmatrix} 3/40 \\ 1/20 \\ 1/16 \\ 1/14 \\ 1/16 \end{bmatrix}$. Our effort to get a smaller game pays, because with can solve this game: the value of game is $v = 1/14$ and the optimal strategy is $[p_1, p_2, p_3, p_4, p_5]^T = [0, 0, 0, 1, 0]^T$.

This translates to $\max(1/v - 9) = 5$ at $x_1 = x_2 = x_3 = x_5 = 0, x_4 = 2, x_6 = 2x_4 - 3 = 1$.

12. We write our LP in a canonical form $-Ax \le b, x \ge 0, cx \to \min$ corresponding to the standard row tableau (13.4) with

$x = [x_1, x_2, x_3, x_4, x_5, x_6, x_7]^T$,

$c = [-1, -2, -4, -1, -1, -1, 0], b = [-5, -6, 7]^T$,

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & -2 & 1 \\ 3 & 1 & 1 & 2 & 1 & 1 & -1 \\ -3 & -1 & -1 & -2 & -1 & 1 & 0 \end{bmatrix}$$

Then

$$M = \begin{bmatrix} 0 & -A & -b \\ A^T & 0 & -c^T \\ b^T & c & 0 \end{bmatrix}$$

is our payoff matrix (see page 219 of the textbook).

13. We take advatage of the fact that all coefficients in the objective function and all right-hand parts of constraints are positive. We set $v = 1/(x_1 + 2x_2 + x_3 + x_4 + x_5 + 3x_6 + x_7 + x_8) > 0$ on the feasible region (because the point where all $x_i = 0$ is not a feasible solution) and $p_1 = x_1 v, p_2 = 2x_2 v, p_3 = x_3 v, p_4 = x_4 v, p_5 = x_5 v, p_6 = 3x_6 v, p_7 = x_7 v, p_8 = x_8 v$. Our constraints take the form

$3p_1 + p_2/2 + p_3 + 2p_4 + p_5 + p_6/3 + p_7 - 3p_8 \ge v$,

$3p_1/5 + p_2/10 + p_3/5 + 2p_4/5 - p_5/5 + p_6/15 + p_7/5 + 3p_8/5 \ge v$,

$3p_1 + p_2/2 + p_3 + 2p_4 + p_5 - p_6/3 + p_7 + p_8 \ge v$.

Other constraints are: $p_i \ge 0$ for all i and $\sum_{i=1}^{8} p_i = 1$. The minimization of $1/v$ is equivalent to the maximization of v.

Thus, we obtain the LP for the row player, with the payoff matrix $\begin{bmatrix} 3 & 3/5 & 3 \\ 1/2 & 1/10 & 1/2 \\ 1 & 1/5 & 1 \\ 1 & -1/5 & 1 \\ 1/3 & 1/15 & -1/3 \\ 1 & 1/5 & 1 \\ -3 & 3/5 & 1 \end{bmatrix}$.

At the position (1,2), we have a saddle point, so the value of game is 0.6 and an optimal strategy for the row player is
$[p_1, p_2, p_3, p_4, p_5, p_6]^T = [1, 0, 0, 0, 0, 0]^T$.
This translate to min $= 5/3$ at $x_1 = 5/3$, the other $x_i = 0$.

§21. Other Methods

T!. *References*

Here is the reference for the fictitious play method:

MR0056265 (15,48e) Brown, George W. Iterative solution of games by fictitious play. Activity Analysis of Production and Allocation, pp. 374–376 Cowles Commission Monograph No. 13. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951. 90.0X T.C. Koopmans (Editor), Ch. XIV

See http://en.wikipedia.org/wiki/Fictitious_play for more details. In particular, ".. modern usage involves the players updating their beliefs simultaneously. Berger goes on to say that Brown clearly states that the players update alternatingly. ."

T2. *Symmetric games*

If the payoff matrix A for a matrix game is skew-symmetric, i.e., $A = -A^T$ as in Example 19.2, then the game is symmetric in an obvious sense. This symmetry implies that the value of game is 0 and that the transpose give an (1-1)-correspomdence between his and her optimal strategies .

A more general symmetry may involve a permutation of player's choices. In Example 19.1, the payoff matrix $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ stays the same after switch the players, the payoffs, and the choices for a player:

$$A = -A^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A^T.$$

This hidden symmetry implies that the value of game is 0 and that we have the following (1-1)-correspomdence between his optimal strategies p and her optimal strategies q:

$$p = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} g^T \text{ or } q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} p^T.$$

Another hidden symmetry in this game is that the matrix A stays the same after switching H and T for both players:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This implies that his mixed strategy $p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ is optimal if and only if $\begin{bmatrix} p_2 \\ p_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} p$ is optimal and her mixed strategy $q = [q1, q2]$ is optimal if and only if $[q_2, q_1] = q \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is optimal. Since the set of optimal strategies is convex, the symmetry implies that her mixed strategy [1/2, 1/2] is optimal.

1. The first row and column are dominated. The optimal strategy for the row player is $[0, 0.5, 0.5]^T$. The optimal strategy for the column player is $[0, 0.25, 0.75]$. The value of the game is 2.5.

2. The second row is dominated by the first row, and the second column is dominated by the third column. So we obtain a 2×2 game $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ which can be easily solved using slopes: optimal strategies are $[0.4, 0.6]^T, [0.4, 0.6]$, and the value of game is 1.2. For the original game, the answer is: optimal strategies are $[0.4, 0, 0.6]^T, [0.4, 0, 0.6]$, and the value of game is 1.2.

3. The optimal strategy for the row player is $[0, 0.4, 0, 0.6]^T$. The optimal strategy for the column player is $[0, 0.4, 0.6]$. The value of the game is 2.8.

5. An optimal strategy for the row player is $[1/3, 1/3, 1/3]^T$. An optimal strategy for the column player is $[0, 0, 2/7, 3/7, 2/7, 0]$. The value of the game is 0.

6. Using slopes, the value of game is 15/4.

7. The value of the game is 0 because the game is symmetric.

8. There is a saddle point at the position (1, 3). The value of game is 1, and optimal strategies are $[1, 0, 0, 0, 0]^T, [0, 0, 1, 0, 0]$.

9. The first two columns and the first row go by domination. It is easy to solve the remaining 2×2 matrix game. The value of game is 11/7, and optimal strategies are $[0, 5/14, 9/14]^T, [0, 0, 5/7, 2/7]$.

11. There is a saddle point at the position (1, 1). So the value of game is 0.

The last column is dominated by any other column. After dropping this column, we get a symmetric game. This is another way to see that the value of game is 0.

13. 0 at a saddle point (at position (1, 1)). The row player has also other optimal strategies, e.g., $[1/3, 1/3, 0, 0, 1/3]^T$

Chapter 8. Linear Approximation

§22. What is Linear Approximation?

1. The mean is $-2/5 = -0.4$. The median is 1. The midrange is $-5/2 = -2.5$.

3. The mean x_2 is 5/9. The median x_1 is 0. The midrange x_∞ is $1/2 = 0.5$.

5. (a) $0, 0, 2, 2, 5$.
(b) $1, 2, 9$.
(c) $0, 0, 2, 2, 3$.
(d). Exercise 1.
(e) $1, 2, 2.2, 3, 3$.
(f). Exercise 3.

7. The program

$$|65 - 1.6c| + |60 - 1.5c| + |70 - 1.7c| \to \min$$

can be easily solved by computing slopes. E.g., the slope of the objective function on the interval $60/1.5 = 40 < c < 65/1.6 = 40.625$ is $1.5 - 1.6 - 1.7 = -1.8$ while the slope on the interval $65/1.6 = 40.625 < c < 70/1.7 \approx 42.2$ is $1.6 + 1.5 - 1.7 = 1.4$. Thus, $\min = 1.875$ at $c = 40.625$.

Since $1.875 < 6.25$ the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^1-metric.

The program

$$(65 - 1.6c)^2 + (60 - 1.5c)^2 + (70 - 1.7c)^2 \to \min$$

can be easily solved by differentiation. We obtain

$$1.6(65 - 1.6c) + 1.5(60 - 1.5c) + 1.7(70 - 1.7c) = 0,$$

hence $c \approx 40.6494$, $\min \approx 1.75325$. So the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^2-metric.

The program

$$\max(|65 - 1.6c|, |60 - 1.5c|, |70 - 1.7c|) \to \min$$

can be easily solved by computing slopes. Near $c = 40.625$, the objective function is $\max(70 - 1.7c, 1.5c - 60)$. So $\min = 0.9375$ at $c = 40.625$. So the model $w = ch$ is better than the model $w = ch^2$ in Example 22.7 when we use l^∞-metric.

9. We enter the data $h = [1.6, 1.5, 1.7, 1.8], w = [65, 60, 70, 80]$ to *Mathematica* as

h = { 1.6, 1.5, 1.7, 1.8}
w = {65, 60, 70, 80}

For $p = 1$, the objective function (to be minimized) is

f=Apply[Plus,Abs[w-a*hˆ 2]]

An optimization command is

FindMinimum[f,{a,1,2}]

The answer is min ≈ 7.59259 at $a \approx 24.6914$. For comparison, the model $w = b$ gives min $= 25$ when $65 \leq b \leq 70$ (medians).

For $p = 2$, we enter

f=Apply[Plus,(w-a*hˆ 2)ˆ 2]; FindMinimum[f,{a,1,2}]

and obtain min ≈ 21.0733 at $a \approx 25.0412$. For comparison, the model $w = b$ gives min $= 218.75$ when $b = 68.75$ (the mean).

For $p = \infty$, we enter

f=Max[Abs[w-a*hˆ 2]]; FindMinimum[f,{a,1,2}]

and obtain min ≈ 3.09339 at $a \approx 25.2918$. For comparison, the model $w = b$ gives min $= 10$ when $b = 70$ (the midrange).

So the model $w = ah^2$ gives better l^p-fits for our data than the model $w = b$ for $p = 1, 2, \infty$.

11. We enter data as in the previous exercise, and then we enter

f=Apply[Plus,Abs[w-a*hˆ 3]]; FindMinimum[f,{a,1,2}]

with the answer min ≈ 21.6477 at $a \approx 14.2479$ for $p = 1$;

f=Apply[Plus,(w-a*hˆ 3)ˆ 2]; FindMinimum[f,{a,1,2}]

with the answer min ≈ 167.365 at $a \approx 14.8198$ for $p = 2$;

f=Max[Abs[w-a*hˆ 3]]; FindMinimum[f,{a,1,2}]

with the answer min ≈ 8.68035, at $a \approx 15.2058$ for $p = \infty$.

13. These are not linear approximations. Taking log of both sides, we obtain a model $\log_2(F_t) = ct$. For this model,

the best l^1-fit is for $c \approx 0.680907$ (with min ≈ 9.5),
the best l^2-fit is for $c \approx 0.680096$ (with min ≈ 2.8),
the best l^∞-fit is for $c \approx 0.671023$ (with min ≈ 0.67).

The limit value for c when we take more and more terms of the sequence is $\log_2(\sqrt{5} + 1) - 1 \approx 0.694242$.

§23. Linear Approximation and Linear Programming

1. min $= 0$ at $a = -15, b = 50$ for $w = a + bh$ and min ≈ 19 at $a \approx 25.23$ for $w = ah^2$

2. $x + y + 0.3 = 0$

3. $a = 0.9, b \approx -0.23$

5. We enter the data in *Mathematica*:

$h = \{1.5, 1.6, 1.7, 1.7, 1.8\}; w = \{60, 65, 70, 75, 80\}$

Then we enter

FindMinimum[Apply[Plus,Abs[w-a*hˆ 2]],{a,1,2}]

The answer is min ≈ 10.1367 at $a \approx 25.3906$.

7. We enter the data in *Mathematica* as in Exercise 5. Then we enter

FindMinimum[Apply[Plus,Abs[w-a*h-b]],{a,1,2}, {b,1,2}]

The answer is min ≈ 13.7647 at $a \approx 40.5882, b \approx 1$.

9. We enter the data in *Mathematica* as in Exercise 5. Then we enter FindMinimum[Max[Abs[w-a*hˆ 2]],{a,1,2}].

The answer is min ≈ 3.09339 at $a \approx 25.2918$.

11. We enter the data in *Mathematica* as in Exercise 5. Then we enter

FindMinimum[Max[Abs[w-a*h-b]],{a,1,2}, {b,1,2}]

The answer is min ≈ 3.72727 at $a \approx 41.8182, b \approx 1$.

12. We get the least squares fit to p_n by $an + b$ when $a \approx 5.53069, b \approx -37.9697$ with min ≈ 11923 being the minimal value for

$$\sum_{i=1}^{100}(p_n - an - b)^2, \text{ while } \sum_{i=1}^{100}(p_n - n\log(n))^2 \approx 161206.$$

13. This is not a linear approximation. Taking log of the both sides, we get a linear model $\log_2(F_n) = \log_2(a) + bn$. The least squares fit is

min ≈ 0.269904 at $b \approx 0.693535, \log_2(a) \approx -0.443517$,
so $a \approx 0.74$.

§24. More Examples

T1. *Why* 95%?

The value for which P = .05, or 1 in 20, is 1.96 or nearly 2 ; it is convenient to take this point as a limit in judging whether a deviation is to be considered significant or not [Fisher R.A . Statistical Methods for Research Workers,Oliver and Boyd, Edinburgh, 1925]. A more precise value is 1.95996 39845 40054 23552... .

If one in twenty does not seem high enough odds, we may, if we prefer it, draw the line at one in fifty (the 2 per cent. point), or one in a hundred (the 1 per cent. point). Personally, the writer prefers to set a low standard of significance at the 5 per cent point, and ignore entirely all results which fail to reach this level. A scientific fact should be regarded as experimentally established only if a properly designed experiment rarely fails to give this level of significance [Fisher R.A. (1926),"The Arrangement of Field Experiments," Journal of the Ministry of Agriculture of Great Britain, 33, 503-513].

Sir Ronald Aylmer Fisher (17 February 1890 – 29 July 1962), who published as R.A. Fisher, was an English statistician, and biologist.

1. The model is $w = ah + b$, or $w - x_2 = a(h - 1988) + b'$ with $b = x_2 + -1988a$ and $x_2 = 37753/45 \approx 838.96$. Predicted production P in 1993 is $x_2 + 5a + b'$.

For $p = 1$, we have $a \approx 16.54, b' \approx 31, P \approx 953$.

For $p = 2$, we have $a \approx 0, b' \approx 32, P \approx 871$.

For $p = \infty$, we have $a \approx 17.59, b' \approx 32, x_5 \approx 959$.

So in this example l^∞-prediction is the best.

3. $a = \$4875, b = \$1500$

4. We enter data for 1900-1998 in *Mathematica* with t replaced by $t - 1990$:

```
t = {0, 1, 2, 3, 4, 5, 6, 7, 8};
x = {421, 429, 445, 449, 457, 460, 481, 503, 514};
y= {628, 646, 764, 683, 824, 843, 957, 1072, 1126}
```

To get the best l^1-fit, we enter

```
FindMinimum[Apply[Plus,Abs[y-a*t-b*x-c]],
{a,1,2}, {b,1,2}, {c,1,2}]
```

with response

```
{280.599, {a -> 47.7205, b -> 1.39226, c -> 1.}}
```

Then we plug these values for a, b, c to $1070-9a-523b-c$ (to get the difference between actual value 1070 and the prediction) and obtain ≈ -89.
To get the best l^2-fit, we enter

FindMinimum[Apply[Plus,(y-a*t-b*x-c)^2],
{a,1,2}, {b,1,2}, {c,1,2}]

with response

{8275.95, {a -> -0.0169939, b -> 5.63814, c -> -1767.27}}

hence $1070 - 9a - 523b - c \approx -111$.
To get the best l^∞-fit, we enter

FindMinimum[Max[Abs[y-a*t-b*x-c]],
{a,1,2}, {b,1,2}, {c,1,2}]

with response

{8275.95, {a -> 90.521, b - > 1.09929, c -> 1.}}

hence $1070 - 9a - 523b - c \approx -321$.

So the l^1-fit gave the best prediction.

T2. *The least-squares*

The reduction of finding the least-squares solutions for $Ax = b$ to finding solutions for $A^T Ax = A^T b$ is done using completion of squares.

In statistics, the favorite way to solve the system is to make columns of A orthogonal to each other by column addition operations (going from left to right).

This is called the Gram-Schmidt process, and it works if the columns of A are linearly independent. The operations do not change the column space but they change variables x. For the new matrix A, the matrix $A^T A$ is diagonal so the system $A^T Ax = A^T b$ splits into independent equations, each having only one variable, so the system can be easily solved.

Computationally, this method is not better than what we did in §6.

T3. *Linear approximation with one unknown*

This is about best approximate solutions to the system $Ax = b$ of m linear equations for n unknowns in the case when $n = 1$. So our system has the form $a_i x = b_i$ for $i = 1, \ldots, m$. In §22, we considered the case when $a_i = 1$ for all i.

For the least-squares fit, we have the mathematical program

$$(a_1 x - b_1)^2 + \cdots + (a_m x - b_m)^2 \to \min$$

with the optimal solutions x being the solutions of the equation $A^T Ax = A^T b$,

Here
$A^T A = a_1^2 + \cdots + a_m^2$
and
$A^T b = a_1 b_1 + \cdots + a_m b_m$.

If $A \neq 0$, the optimal x is unique. If $A = 0$, every x is optimal. As in the general case, Ax is unigue.

For the LAD fit, we have the mathematical program

$$|a_1 x - b_1| + \cdots + |a_m x - b_m| \to \min .$$

When all $a_i = 1$, the optimal solutions are the medians of the numbers b_i. So when all a_i are positive integers, we know that the optimal x are the medians of the set of the numbers b_i/a_i with each b_i/a_i repeated $|a_i|$ times. Therefore the solving the program with rational a_i can be reduced to finding medians for a set of numbers with repetitions (multi-set).

For real a_i it is the medians of a weighted set of numbers. For any number x let L be the total weight strictly on the left, C the wight at x, and R the total weight strictly on the right. Then x is a median if and only if $L + C \geq R$ and $L \leq C + R$. The left slope at x is $L - C - R$ and the right slope at x is $L + C - R$.

Anyhow, the graphical method with one unknown x is better than the simplex method with the standard row tableau having 3 variables on top and $2m$ variables at the right margin.

The objective function is convex and piece-wise linear. The corners are at $x = b_i/a_i$ with $a_i \neq 0$. A corner is optimal, if and only if the left slope is ≤ 0 and the right slope is ≥ 0. If both slopes are positive, we go to the left for the min. If both slopes are negative, we go to the right for the min. This allows us to avoid computing all the slopes of the objective function. Using *bisection* (cf, p. 262 of the textbook), it suffices to check $\leq \log_2 k$ corners where k is the number of corners.

For the uniform fit, we have the mathematical program

$$\max(|a_1 x - b_1|, \ldots, |a_m x - b_m|) \to \min .$$

The objective function is convex and piece-wise linear. It has up to $2m-1$ corners. The graphical method is still better than the simplex method The standard row tableau has $m + 2$ variables on top and $2m$ variables at the right margin.

T4. *LAD with* 3 *unknowns*

Check whether A = [0,0,0] or B = [1,1,1] is optimal for $f = |x| + |y-1| + |z| + |x+y+z-3| \to \min$.

Solution. At A, $f = 4$. Near A, $f = |x|+1-y+|z|+3-x-y = |x|+|z|-x-2y+4$. Setting $x = z = 0$ and $y = 0.01$, we get a value less than 4. So A is not optimal.

At B,$f = 2$ (so A is not optimal). Near B,

$f = x + |y-1| + z + |x+y+z-3|$.

Since $|y-1| \geq y-1$ and $|x+y+z-3| \geq 3-x-y-z$ for all x, y, z, we have $f \geq 2$ near B. Since f is convex, B is optimal.

T5. *Uniform approximation with* 3 *unknowns*

Check whether A = [0,0,0] or B = [1,1,1] is optimal for $f = \max(|x|, |y-1|, |z|, |x+y+z-3|) \to \min$.

Solution. At A,$f = \max(0,1,0,3) = 3$. Near A, $f = 3-x-y-z$. Setting $x = y = 0$ and $z = 0.01$, we see that A is not optimal.

At B,$f = \max(1,0,1,0) = 1$ (so A is not optimal). Near B, $f = \max(x, z)$. Taking $y = 1$ and $x = z = 0.99$, we see that B is not optimal.

T6. *Euclidean plane and Pythagorean theorem*

Euclid defined his plane by a list of axioms. By a modern definition, it consists of pairs of numbers and the distance $d(a,b) \geq 0$ between two pairs $a = [a1, a2], b = [b1, b2]$ is defined by

$$d(a,b)^2 = ||a-b||_2 = (a1-b1)^2 + (a2-b2)^2.$$

The Pythagorean theorem is about 3 distances between 3 points a, b and c in the plane. It says that when $a-c$ and $b-c$ make a right anngle, i.e., the dot product of $a-c$ and $b-c$ is 0, then

$$d(a,c)^2 + d(b,,c)^2 = d(a,b)^2.$$

The fact was known and previously utilized by the Babylonians and Indians Because of the secretive nature of his school and the custom of its students to attribute everything to their teacher, there is no evidence that Pythagoras himself worked on or proved this theorem.

T7. *Shape of Earth*

Without the framework of Euclidean plane, this Pythagorean theorem is not true. For example consider the following geometric triangle: c is the north pole, a and b are on the equator at longitude 0^o and 90^o. Then all three angles of the triangle are 90^o. Every side is about 10,000 km long. So the surface of Earth is not an Euclidean plane.

A sphere is a better approximation. A more precise value for the length of ab is 10,016 km while ac and bc are about 10,001 km. So it is not exactly a sphere.