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On Sums of Three Integral Cubes

by W. CONN & L. N. VASERSTEIN

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ON SUMS OF THREE INTEGRAL CUBES

W Conn and L. N Vaserstein

Abstract Using computers, we found new representations of natural numbers $t < 1000$ as sums of three integral cubes. In particular, we found the first representations for $t = 39, 84, 556, 870$, and 960. We also found the first three primitive representations for $t = 80$ and the first representation for $t = 2$ which does not belong to a known polynomial family.

In 1955, at Mordell's suggestion, Miller and Woollett [5] investigated the integral solutions of the equation

$$x^3 + y^3 + z^3 = t \tag{1}$$

with

$$1 < t \leq 100, |z| \leq |y| \leq |x| \leq 3200 \tag{2}$$

using the EDSAC computer at Cambridge University. They found 447 primitive (i.e. with $\mathrm{GCD}(x, y, z) = 1$) solutions of (1) in the region (2).

Reduction modulo 9 shows that (1) has no integral solutions with $t = \pm 4 \bmod 9$. For all other t no obstructions to solving (1) are known. Solutions for all other t in the interval $1 < t \leq 100$ were found except for

$$t = 30, 33, 39, 42, 52, 74, 75, 84, 87 \tag{3}$$

For $t = 12, 16, 44, 48, 51$, and 66, only one primitive solution per t was found, e.g.,

$$12 = -11^3 + 10^3 + 7^3, \qquad 16 = 1626^3 - 1609^3 - 511^3,$$

$$51 = -796^3 + 659^3 + 602^3, \qquad 66 = 4^3 + 1^3 + 1^3$$

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For $t = 0$, it had been known that all integer solutions of (1) come (up to permutation of x, y, z) from the polynomial solution $u^3 - u^3 + 0^3 = 0$. For $t = 1$ and 2, the following polynomial solutions of (1) had been known

(4) $$(9u^4)^3 - (9u^4 - 3u)^3 - (9u^3 - 1)^3 = 1,$$

(5) $$(6u^3 + 1)^3 - (6u^3 - 1)^3 - (6u^2)^3 = 2.$$

The identity (4) accounts for 9 of the 23 integral solutions of (1) in the region (2) found in [5], and the identity (5) accounts for all 9 solutions. Using (4) and (5) one derives infinitely many integral solutions of (1) in the case when $t = k^3$ or $2k^3$ with an integer k

It is well-known (S. Ryley, 1825 [1, Ch. XXV, p.726], see also [4]) that the Diophantine equation (1) has infinitely many rational solutions for any rational t So 'solution' will mean 'integral solution', and x, y, z, t, k below are integers.

Mordell [6, p.139] pointed out that he knew no answer to the following question. are there other integer t for which (1) 'has an infinity of integer solutions x, y, z' It is also unknown whether (1) has infinitely many primitive solutions when $2k^3, k \neq 0, \pm 1$ On the other hand, when t is a cube, (1) has infinitely many 'trivial' solutions with $x + y, x + z$, or $y + z = 0$. Moreover, it is known [1], [8], [9] that there are infinitely many primitive nontrivial solutions of (1) as well as infinitely many polynomial solutions [3] No polynomial solutions are known for $t \neq k^3, 2k^3$ [8], [10]

Among queries posed in [5], were the questions whether (1) has a solution with t from the list (3) and whether (1) has a solution with $t = 2$, not given by (5).

In 1964, the solutions of (1) with

(6) $$t \text{ not a cube}, 1 < t \leq 1000, |y| \leq |x| \leq 65536$$

were searched for using I. B. M. STRETCH and MANIAC II computers [2]. One of the 'excluded' targets in (2) was represented as a sum of three cubes. Namely,

$$87 = 4271^3 - 4126^3 - 1972^3.$$

Also, the first primitive solution for $t = 96$ was found in [2]:

$$96 = -15250^3 + 13139^3 + 10853^3$$

This left $t = 24$ and 80 the only values of t in the interval $1 \leq t \leq 100$ for which only 'derived' (i.e., not primitive) solutions of (1) were known.

The conclusion of [2] was that their results made it unlikely that every integer $\neq \pm 4 \pmod 9$ was the sum of 3 integral cubes. Moreover, in the opinion of the authors of [2], it was rather unlikely that all t in (3) would turn out to be expressible as sums of three cubes. All solutions of (1) with $t = 2$ found in [5] and [2] belong to the parametric family (5).

A search for solutions of (1) in the region $|x| \leq |y| \leq 100,000$ with some t in (3) gave nothing [9].

Nevertheless, we were not discouraged by these results and carried out further computations, using Mathematica at Sun 4 and NeXT workstations, as well as other software and hardware. We found a solution of (1) with $t = 39$, namely,

$$39 = \quad 159380^3 + 134476^3 + 117367^3,$$

a solution with $t = 84$, namely,

$$84 = 41639611^3 \quad 41531726^3 - 8241191^3,$$

three primitive solutions with $t = 80$, namely,

$$80 = \quad 279817^3 + 262880^3 + 155257^3 = -112969^3 + 103532^3 + 69241^3$$

and a solution with $t = 2$ which is not included in (5), namely,

$$2 = -3528875^3 + 3480205^3 + 1214928^3$$

Here is the list of all t in the interval $1 \leq t \leq 100$ for which no primitive solutions of (1) are known.

$$t = 24, 30, 33, 42, 52, 74, 75. \tag{7}$$

Here is the list of t in the interval $1 \leq t \leq 100$ for which only one primitive solution of (1) with $|x| \geq |y| \geq |z|$ is known.

$$t = 12, 16, 39, 51, 84, 87, 96. \tag{8}$$

The solutions were all given above.

In Table 1, for all t in the interval $0 \leq t \leq 100, t \neq \pm 4(\bmod 9), t \neq k^3, k^3$, for which at least two primitive solutions of (1) with $|x| \geq |y| \geq |z|$ are known, except $t = 80$ (for which the 3 known solutions are given above), we give the number MW of primitive solutions listed in [5], the number GLS of new solutions found by the authors of [2], the number CV of new solutions we found, and the total number N = MW + GLS + CV of solutions known. Also listed in Table 1 are the two solutions with maximal $|x|$

Table 1. The number of known integral solutions of (1) and some solutions

t	MW	GLS	CV	N	x, y, z	x, y, z
3	2	0	0	2	1, 1, 1	-5, 4, 4
6	4	1	1	6	-60355, 60248, 10529	-2593669, 2441147, 1426148
7	3	0	3	6	-6150123, 6137561, 1124657	-5575582, 5547575, 1374870
9	2	1	1	4	-11329, 11305, 2097	-71521992, 71463217, 9659294
10	4	0	2	6	301471, -294038, -125509	298197006, -298124987, -26780787
11	4	1	1	6	6628, -5973, -4274	501483, -501026, -70100)
15	3	0	3	6	1562828, -1562761, -78886	-31430185, 31278578, 7646792
17	5	2	2	9	330502, -296407, -215872	763403, -677757, -511173
18	4	3	5	12	29667623, -29533949, -7056540	3107173978,-3107114991,-119545570
19	4	0	3	7	549195,-498390, -347186	4527187, -4086798, -2905548
20	7	1	6	14	3645939, -3431087, -2006066	9348001, -9161277, -3633722
21	4	1	3	8	-106358, 106333, 9466	12124, -10931, -7808
25	3	1	0	4	-15964, 15942, 2561	-862850, 714469, 652408
26	4	1	1	6	-63154, 62709, 17421	139643, -132288, -74169
28	5	1	2	8	500752, -440941, -341519	443280029, -443049953, -51369794
29	4	3	5	12	-1639004, 1604581, 647628	-59565612, 59429101, 11317786
34	8	2	3	13	804494, -802695, -151615	2385818, -2383221, -353833
35	3	7	2	12	256817, -231035, -166387	8564556, -8549692, -1483557
36	4	2	5	11	21554787, -21184007, -7978924	-46498000, 46496077, 2319087
37	3	0	2	5	-155924, 154165, 50246	-5148254, 5139542, 884317
38	2	1	0	3	-27 25, 16	12205, -10940, 7983
43	5	3	5	13	1668803, -1647658, -558738	-14191393, 14154155, 2820585
44	1	0	3	4	-264878, 264623, 37709	1482566, -1482369, -109107
45	3	1	4	8	1136437, -940266, -860158	-1674692, 1342485, 1315652
46	3	1	3	7	-558258, 495645, 373777	598825, -597584, -110035
47	4	0	3	7	-213705, 209650, 81688	-392291, 391641, 66913
48	1	1	0	2	31, -26, -23	3991, -3950, -1247
51	1	0	1	2	-796, 659, 602	886475, -885556, -129352
53	5	0	6	11	1924978, -1922482, -302611	160529381, -160267130, -27252892
54	3	2	0	5	4459, -3613, -3462	12437, -11375, -7674
55	11	3	4	18	-38424926, 38198927, 9983872	452034271, -451993693, -29190299
56	3	0	6	9	-11151164, 11095150, 2749740	27129289, -27128828, -1005921
57	7	3	4	14	1256119, -1220489, -547277	-103473047, 103393096, 13690564
60	2	1	1	4	-8233, 7061, 5906	-1146811 1036031, 734480
61	2	0	0	2	5, -4, 0	-966, 845, 668
62	7	3	6	16	30326285, -30255846, -5787903	-38123721, 38047638, 6917951
63	5	3	4	12	-868399, 867133, 141945	10555335, -10477547, -2955229
65	5	3	0	8	-25248, 24193, 12460	47425, -43629, -28691
66	1	0	1	2	4, 1, 0	1619125, -1619036, -88787
69	4	0	2	6	-1213102, 1209029, 261692	-18037816, 18036296, 1140509
70	5	1	2	8	-2858077 2827923, 900896	3322634, -3317619, -549415
71	10	3	3	16	-6681513, 6502032, 2860160	-8609744, 8588678, 1671887
72	2	0	0	2	-10, 7, 9	28, -27, -13
73	4	3	1	8	56729, -50552, -37652	705811 -703771, -144863
78	2	0	0	2	-55, 53, 26	78, 2123, -2080, -829
79	3	0	0	3	74, -66, -49	711, -706, 196
81	3	0	3	6	-294596, 286361 127746	355419224, -355399661, -19498665
82	2	1	2	5	-390539, 344669, 265048	167797 -152931, -104700
83	13	3	6	22	-991895, 962193, 439801	6817294, -6816464, -487293
88	3	2	3	8	1156223, -1156023, -92908	-46266712, 46225115, 6438381
89	3	0	4	7	-12534879, 12526638, 1571636	26743320, -26743052, -831561
90	14	5	10	29	-66708371, 66702677, 4235982	541378689, -541364578, -23150303
91	5	2	6	13	-222677, 209968, 121298	-6817294, 6816464, 487293
92	8	3	5	16	-7439189, 7362352, 2328537	-19488167 19261291, 6345444
93	2	0	0	2	7 -5, -5	253, -248, -98
97	8	1	7	16	-11656809, 11656805, 117701	-98258218, 98076953, 17369628
98	4	0	1	5	2391, -2101, -1638	-301423, 293034, 130521
99	9	9	11	29	3916074, -3672925, -2190200	6097855, -5802483, -3154129
100	3	0	2	5	768040, 688993, 501307	1141317774, -1141296093, -43920623

Except for the representations of 87 and 96 quoted above, solutions of (1) are not explicitly listed in [2], but their number is given for each t, and we found exactly so many solutions in the region (6) to account for these numbers. So the results of [2] are confirmed by our computations for $t \leq 100$. The method of [2] allowed the authors to obtain solutions for larger t in the same sweep, without spending substantial additional time. In [2], the numbers of solutions of (1) are given for all t in the interval $1 \leq t \leq 999$ which are not cubes.

We searched several regions including the regions

(9) $|x| \geq |y| \geq |z|, |x+y| \leq 49, |z| < 1{,}000{,}000,$

$$1 < t \leq 1{,}000;$$

(10) $|x| \geq |y| \geq |z|, 50 \leq |x+y| \leq 200, |z| \leq 20{,}000||x+y|,$

$$1 \leq t \leq 1{,}000;$$

(11) $|x| \geq |y| \geq |z|, 201 \leq |x+y| \leq 1{,}999, |z| \leq 20{,}000|x+y|,$

$$1 \leq t \leq 100, t \text{ not a cube};$$

(12) $|x| \geq |y| \geq |z|, 2{,}000 \leq |x+y| \leq 75{,}000, |z| \leq 2{,}998|x+y|,$

$$1 \leq t \leq 100, t \text{ not a cube};$$

(13) $|x| \geq |y| \geq |z|, 75{,}001 \leq |x+y| \leq 446{,}000, |z| \leq 298|x+y|,$

$$1 \leq t \leq 100, t \text{ not a cube}.$$

There are 520 primitive solutions of (1) with $|x| \geq |y| \geq |z|, 1 \leq t \leq 100, t$ not a cube, in the regions (9)–(13), excluding exactly 56 solutions coming from (5) with $u = 0, 1, \ldots, 55$. This includes 341 solutions known previously (from [5] and [2]). Of these 520 solutions, 246 are in the region (9), 48 are in the region (10), 76 are in (11), 111 are in (12), and 39 are in (13).

The 12 maximal ratio $-z/(x+y)$ obtained rounded to the closest integers are 29,425; 19,483; 11,921, 9,934; 6,118; 2,159; 2,027; 2,026; 1,641; 1,317; 1,206; 1,177 The maximal values of $|x|, |y|, |z|, s$ among our solutions are

$$3,107,173,978; \quad 3,107,114,991, \quad 119,545,570; \quad 440,389$$

respectively. Figure 1 below plots the points

$$(\log(|x+y|+1), \log(|z|+1) - \log(|x+y|+1))$$

for our 520 solutions and shows the upper bounds for the regions (9)–(13) of search.

Computers also produced many solutions with $t = 1, 8, 27, 64$ and with $t > 100$, too many to show here. We give here only representations for numbers for which it was unknown whether they were sums of three integral cubes ($t = 556, 870$) or no primitive representation was known ($t = 960$):

$$556 = -1379083^3 + 1379046^3 + 59543^3,$$

$$870 = -420598^3 + 420449^3 + 42917^3,$$

$$960 = -1753258^3 + 1753229^3 + 64427^3$$

Several methods and programs were used. We will describe the methods which were most efficient in regions (9), (10), and (11)–(13). Set $s = x + y$. Then (1) takes the form

$$s(x^2 - xy + y^2) + z^3 = t, \tag{14}$$

hence $z^3 \equiv t \pmod{s}$. Thus, the condition $1 \le t \le 100$, allows us to dismiss many congruence classes of z modulo s (for large s).

Next we rewrite (1) as $(-y+s)^3 + y^3 = t - z^3$ or

$$3s(y^2 - ys) = t - z^3 - s^3, \tag{15}$$

hence $t - z^3 \equiv s^3 \pmod{3s}$. This condition imposes further restrictions on z. Finally we use that $4(y^2 - ys) = (2y - s)^2 - s^2$ and rewrite (15) as

$$v^2 = 3s(4t - 4z^3 - s^3) \text{ with } v = 3s(2y - s) \equiv -3s^2 \pmod{6s}. \tag{16}$$

In our first method, used in the regions (9) and (10), we fixed s first ($1 \leq s \leq 200$). Then we ran z from $-s$ to -1,000,000 (resp., to 20,000s) to obtain solutions in region (9) (resp., (10)). For each s we compute v as the integer closest to $(-12z^3 - 3s^4)^{(0.5)}$. If $v \equiv -3s^2 \pmod{6s}$ and $|v^2 + 12z^3 - 3s^4| \leq 1,000$, we obtain a solution. This program misses some solutions with small $|x|, |y|, |z|$, but they can be easily found by other methods (e.g., looking up [5]).

Our second program, used for $s \geq 201$, also fixes s first. Then it tries all integers c in the interval $-s \leq c < 2s$ and selects such that $c^3 \equiv t(\bmod s)$ with $|t| \leq 100$. We fix the target t at this point and dismiss all c which do not satisfy the condition $t - c^3 \equiv s^3 \pmod{3s}$. To save running time, we dismiss all targets s which are cubes (there are too many solutions for them) as well as any $t \equiv \pm 4 \pmod 9$ (for such t, there are no solutions). Now we run z along the arithmetic progression $z = c - 3sw$, and proceed as in the end of the previous paragraph. The intervals for we used w were: $1 \leq w \leq 2,667$ when $201 \leq s \leq 2,000; 1 \leq w \leq 1,000$ when $2,001 \leq s < 75,000; 1 \leq w \leq 1,000$ when $75,001 \leq s < 446,000$. The condition $|y| \geq |z|$ is satisfied automatically for $w \geq 2$. Allowing $w = 1$, we obtained many solutions for the second (or even third) time with a different value of s (with x, y, z permuted and signs changed). In runs with Mathematica we usually included $w = 0$, which resulted in many solutions of (1) in Gauss integers. Here are a few examples:

$$
\begin{aligned}
3 &= -(15982 + 4010i)^3 - (15982 - 4010i)^3 + 18779^3 \\
&= -(1 + i)^3 - (1 - i)^3 - 1^3 = (2 + i)^3 + (2 - i)^3 - 1^3, \\
30 &= (95 + 214i)^3 + (95 - 214i)^3 + 290^3 \\
&= (95 + 514i)^3 + (95 - 514i)^3 + 530^3
\end{aligned}
$$

If we had increased the set of possible targets t , the second method would have required more computational time, so we did not use it to solve (1) with $t > 100$. The reason we decreased the upper bound for w for runs with larger s was to save running time. We believe that we missed some solutions as a result of this.

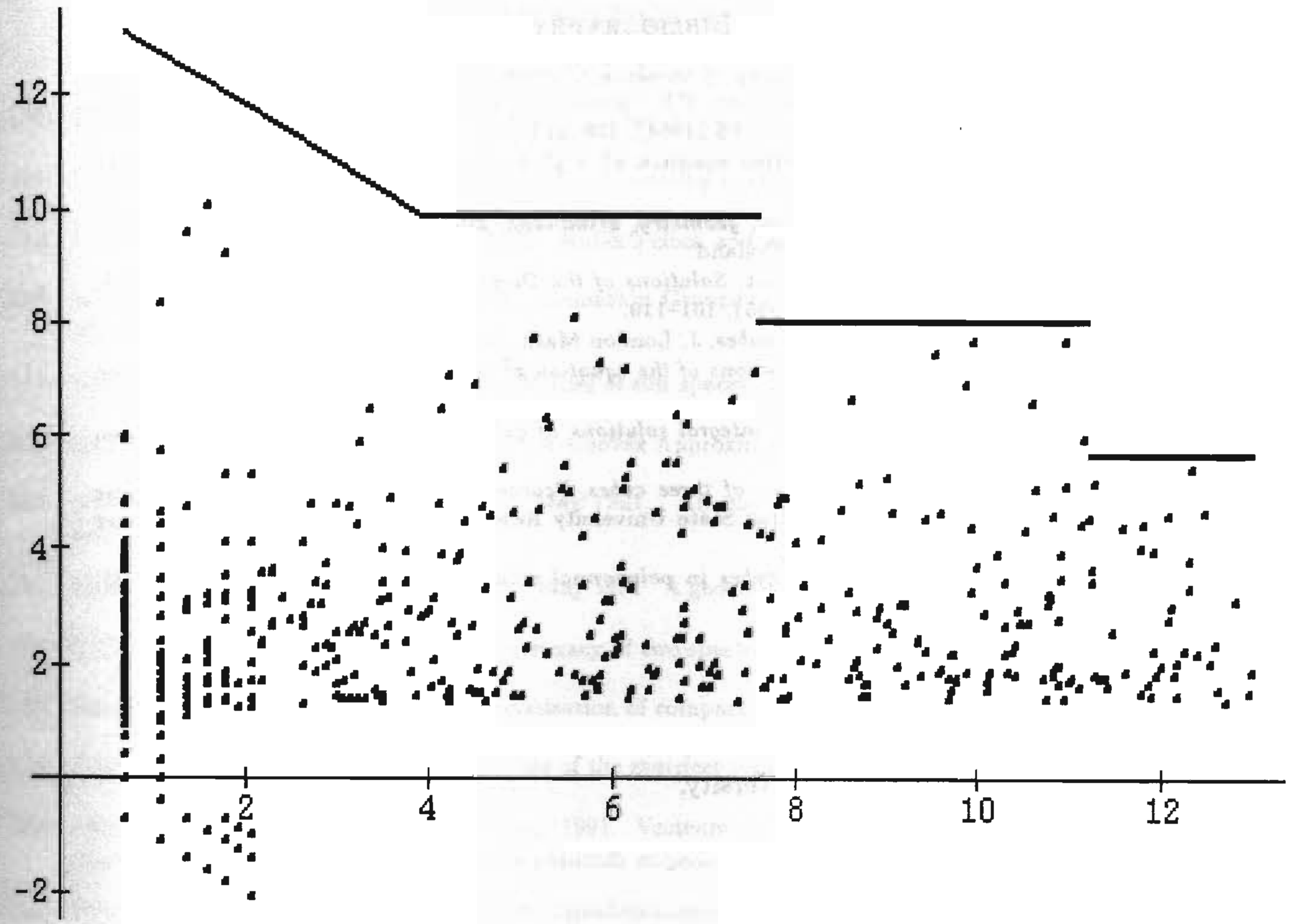

Figure 1. The points $(\log(s+1), \log(|z+1|) - \log(s+1))$ for 520 primitive solutions (x, y, z) with $x \geq |y| \geq |z|$ and $s = x + y$

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Department of Mathematics,
The Pennsylvania State University,
University Park, PA 16802.

E-mail addresses: bill@math.psu.edu and vstein@math.psu.edu

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