PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE

MASTER THESIS

Characterization of uniform hyperbolicity for fiber-bunched cocycles

Author:
Renato Velozo Ruiz

Supervisor:
Professor Jairo Bochi

A thesis submitted in fulfillment of the requirements for the degree of Master in Mathematics in the Faculty of Mathematics of the Pontifical Catholic University of Chile.

Jury:
Professor Godofredo Iommi (Pontifical Catholic University of Chile)
Professor Radu Saghin (Pontifical Catholic University of Valparaíso)

July 2018
Santiago, Chile

Acknowledgements

I would like to thank to my supervisor Jairo Bochi for his encouragements and his continued guidance throughout all this work. Specially for introduce me to this particular problem and for the invaluable time he spent talking about mathematics with me. During this process he helps me learn about mathematics, proofs, typing and life. I thank him for alI that. I am going to be always grateful for his constant support during the previous years.

I would like to thank the professors Godofredo Iommi and Radu Saghin for accept to be readers of this thesis and for their helpful comments.

I would like to thank the professors Duvan Henao and Mariel Sáez for their support, guidance and friendship since my first years at the university. I really appreciate the time they spent talking with me about life and mathematics. I would like to thank the professors Godofredo Iommi, Jan Kiwi and Rafael Benguria for their support and guidance during my application to the PhD. I would like to thank the professors Matias Courdurier, Alberto Montero and Alvaro Cofré for many interesting conversations about mathematics.

I would like to thank to my brother Anibal Velozo for his encouragements since I decided to be a mathematician. I will never forget our interesting conversations about mathematics in the car each day in our way home during my first years at the university.

I would like to thank to all my friends from the PUC. Specially to Sebastian Burgos, Ariel Reyes, Nicolás Alvarado, Aída Aqueveque, Sandra Garrido, Matias Villagra and the rest of my friends of the office M23. They have made me feel at home during all the years I have been here at the PUC. Also, I would like to thank to my friends Adrián Bustos, Camilo Codorniu, Felipe Pastén, Frane Sazunic and my personal trainer Mauricio Pavéz for their company and continuous support. They helped me to get perspective about mathematics and my life.

I would like to thank to the most important people in my life: my family. I would like to thank my brothers Anibal and Renzo for cheer me up each time we spend together. I would like to thank my parents for their inconditional love. Also, I would like to thank to my grandmother Mirna, to my grandfather Juan Milano and to my grandfather Juan Velozo for look after me continuosly. They are a central part of my life.

Thank you to all the people who have helped me throughout this process.

I would like to thank to the Faculty of Mathematics of the Pontifical Catholic University of Chile by the funding I received during my first year at this program. In addition, I would like to thank to CONICYT by funding my second year of this program through the scholarship CONICYT Folio 22180035. This thesis was partially supported by CONICYT PIA ACT172001.

Contents

Chapter 1

Introduction

A linear cocycle is a pair (T, A) where $T : X \to X$ is a homeomorphism defined on a compact metric space X and $A : X \to GL(d, \mathbb{R})$ is a continuous function. In particular, a $SL(2, \mathbb{R})$-cocycle is a linear cocycle where A takes values on $SL(2, \mathbb{R})$. Besides, we are going to use the following notation

$$A^n(x) := A(T^{n-1}x)A(T^{n-2}x)\dots A(Tx)A(x),$$

$$A^{-n}(x) := A(T^{-n}x)^{-1}A(T^{-n+1}x)^{-1}\dots A(T^{-2}x)^{-1}A(T^{-1}x)^{-1},$$

and $A^0(x) := I$ for every $x \in X$ and $n > 0$. Along this thesis we are going to deal specifically with uniformly hyperbolic cocycles.

Definition 1.1. (=Definition 3.1) A $SL(2, \mathbb{R})$-cocycle (T, A) is called *uniformly hyperbolic* if there are constants $c > 0$ and $0 < \lambda < 1$ such that for every $x \in X$ there exist transverse one-dimensional spaces E^s_x and E^u_x in $\mathbb{R}^2$ such that

1. $A(x)E^s_x = E^s_{T(x)}$ and $A(x)E^u_x = E^u_{T(x)}$,
2. $\|A^n(x)v^s\| \leq c\lambda^n\|v^s\|$ and $\|A^{-n}(x)v^u\| \leq c\lambda^n\|v^u\|$,

for every $x \in X$, $v^s \in E^s_x$, $v^u \in E^u_x$ and $n \geq 1$.

Here $\|\cdot\|$ denotes the Euclidean norm. For $SL(2, \mathbb{R})$-cocycles there is a well known characterization of uniform hyperbolicity proved by J.-C. Yoccoz in [Y] (see [BG] and [Z] for related results).

Proposition 1.2. *(=Proposition 3.2) A $SL(2, \mathbb{R})$-cocycle (T, A) is uniformly hyperbolic if and only if there are constants $c > 0$ and $\tau > 0$ such that*

$$\|A^n(x)\| \geq ce^{\tau n}, \quad \text{for all } n \geq 0 \text{ and } x \in X.$$

In the previous proposition $\|\cdot\|$ is the operator norm induced by the Euclidean norm. We proceed to define another main concept in these notes: Lyapunov exponents.

Definition 1.3. (=Definition 3.9) Let (T, A) be a linear cocycle. We define the *upper and lower Lyapunov exponents at a point* $x \in X$ respectively by

$$\lambda_+(x) := \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \quad \text{and} \quad \lambda_-(x) := \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1},$$

whenever the limits exist.

It follows from Kingman's subadditive ergodic theorem that these limits exist for every $x \in \mathcal{R}$, where $\mathcal{R} \subset X$ is a Borel set such that $\mu(\mathcal{R}) = 1$ for any T-invariant probability measure μ. The elements of $\mathcal{R}$ are called *regular points*. By elementary linear algebra, every periodic point is regular. For more details about the properties of Lyapunov exponents see [AB2] and [V].

By Proposition 1.2, for every $SL(2,\mathbb{R})$-cocycle which is uniformly hyperbolic, there is a constant $\tau > 0$ such that

$$\lambda_+(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \geq \tau > 0, \quad \text{for every } x \in \mathcal{R}.$$

In addition, since $\|M\| = \|M^{-1}\|$ for every $M \in SL(2,\mathbb{R})$, we have $\lambda_+(x) = -\lambda_-(x)$ for every $x \in \mathcal{R}$. Hence, there is a uniform gap of 2τ between the Lyapunov exponents. More precisely,

$$\lambda_+(x) - \lambda_-(x) \geq 2\tau \quad \text{for every } x \in \mathcal{R}.$$

In the following, we are going to show that this property characterizes uniform hyperbolicity for an important class of cocycles. Before to state the result, we recall a basic definition.

Definition 1.4. A Borel set $\mathcal{S} \subset X$ is called a *full probability set* if $\mu(\mathcal{S}) = 1$ for every T-invariant probability measure μ.

In particular, the set $\mathcal{R}$ of regular points is of full probability. Furthermore, each periodic point $p = T^n p$ belongs to every Borel set $\mathcal{S} \subset X$ of full probability. In fact, for the T-invariant probability measure μ_p defined by

$$\mu_p = \frac{\delta_p + \delta_{Tp} + \cdots + \delta_{T^{n-1}p}}{n},$$

the periodic point p has positive measure. We proceed to state the main result of this thesis.

Theorem 1.5. *(=Theorem 4.2) Let (T, A) be a $SL(2,\mathbb{R})$-cocycle defined over a transitive subshift of finite type or a transitive Anosov diffeomorphism. Suppose the cocycle satisfies the fiber-bunching condition, and there is a constant $\tau > 0$ and a full probability set $\mathcal{S} \subset \mathcal{R}$ such that*

$$\lambda_+(x) \geq \tau \quad \textit{for every } x \in \mathcal{S}.$$

Then the cocycle (T, A) is uniformly hyperbolic.

See Chapter 3 for the definition of fiber-bunched linear cocycles. Note that Y. Cao proved in [C] a similar characterization of uniform hyperbolicity but assuming a stronger hypothesis. In fact, Y. Cao assumed a continuous invariant splitting in the tangent bundle.

We also show the fiber-bunching condition is necessary for the validity of Theorem 1.5. In fact, we construct a cocycle over a subshift of finite type which has uniform gap between the Lyapunov exponents, nevertheless it is not uniformly hyperbolic. More precisely, we get the following theorem.

Theorem 1.6. *(=Theorem 5.3) There is a $SL(2,\mathbb{R})$-cocycle (T, A) defined over a subshift of finite type such that,*

1. *The cocycle (T, A) is not uniformly hyperbolic.*

2. *There is a set of full probability $\mathcal{S}$ such that $\lambda_+(x) \geq \log 2/2 > 0$ for every $x \in \mathcal{S}$.*

Now, we describe the organization of this thesis. In Chapter 2 we recall some basic definitions and concepts about subshifts of finite type and Anosov diffeomorphisms. In Chapter 3 we recall some basic definitions and concepts about linear cocycles (T, A). More specifically, we are going to study $SL(2,\mathbb{R})$-uniformly hyperbolic cocycles which satisfy the fiber-bunching condition. In Chapter 4 we prove Theorem 1.5. In Chapter 5 we construct the example of Theorem 1.6. Finally, we conclude with Chapter 6 which presents some remarks and questions which remain open.

Chapter 2

Dynamics and Basic Concepts

2.1 Subshifts of finite type

We start recalling some basic definitions and properties of the subshifts of finite type.

Definition 2.1. Let $Q = (q_{ij})$ be a $l \times l$ matrix with $q_{ij} \in \{0, 1\}$. The *subshift of finite type associated to the matrix* Q is a dynamical system $T : X \to X$, where X is the set of sequences

$$\{(\dots, x_{-1}|x_0, x_1, \dots) \in \{1, 2, \dots, l\}^{\mathbb{Z}} : q_{x_n x_{n+1}} = 1 \text{ for every } n \in \mathbb{Z}\},$$

and T is the left-shift map defined by $T((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$.

Moreover, we are going to consider the following metric on X,

$$d(x, y) := \begin{cases} 2^{-N(x,y)} & \text{where } N(x, y) := \min\{|n| \geq 0 : x_n \neq y_n\}, \\ 0 & \text{if } x = y. \end{cases}$$

Note that (X, d) is a compact metric space and T is a homeomorphism.

Definition 2.2. Let $Q = (q_{ij})$ be a $l \times l$ matrix with $q_{ij} \in \{0, 1\}$. The matrix Q is called *irreducible* if for every pair $i, j \in \{1, 2, \dots, l\}$ there is $m_{ij} \geq 1$ such that $(Q^{m_{ij}})_{ij} > 0$.

Proposition 2.3. *Every subshift of finite type associated to an irreducible matrix Q is transitive, in other words it has a dense orbit.*

Proof. It is enough to prove that the set of periodic points $Per(T)$ is dense in X. In fact, as $Per(T) = \{p_n\}_{n\in\mathbb{N}}$ is a countable set, we can easily construct a point $x \in X$ which has a dense orbit.

Claim 2.4. The set of periodic points $Per(T)$ is dense in X.

Let $\alpha = (\alpha_{-k}, \dots, \alpha_k)$ an arbitrary sequence such that $Q_{\alpha_i \alpha_{i+1}} = 1$ for every $i \in \{-k, \dots, k\}$. Since $(Q^{m_{ij}})_{ij} > 0$ for some positive integer $m_{i,j}$ we can extend α to a periodic sequence $\tilde{\alpha} \in X$ of period $2k + 1 + m_{ij}$. In particular, each cylinder has a periodic point. Since the cylinders form a basis for the topology, the set of periodic points $Per(T)$ is dense in X. ■

We define the *local stable set of* $x \in X$ by

$$W^s_{loc}(x) := \{(y_n)_{n\in\mathbb{Z}} \in X : y_n = x_n \text{ for every } n \geq 0\},$$

and the *local unstable set of* $x \in X$ by

$$W^u_{loc}(x) := \{(y_n)_{n\in\mathbb{Z}} \in X : y_n = x_n \text{ for every } n \leq 0\}.$$

The *global stable and unstable manifolds of* x are defined by

$$W^s(x) := \bigcup_{n=0}^{\infty} T^{-n}(W^s_{loc}(T^n x)) \quad \text{and} \quad W^u(x) := \bigcup_{n=0}^{\infty} T^{n}(W^u_{loc}(T^{-n} x)).$$

Proposition 2.5. *The global stable and unstable manifolds of x are characterized by*

$$W^s(x) = \{y \in X : \lim_{n\to\infty} d(T^n x, T^n y) = 0\} \quad \textit{and} \quad W^u(x) = \{y \in X : \lim_{n\to\infty} d(T^{-n} x, T^{-n} y) = 0\}.$$

Proof. Let $y \in W^s(x) = \bigcup_{n=0}^{\infty} T^{-n}(W^s_{loc}(T^n x))$ by definition there exists N such that $T^N y \in W^s_{loc}(T^N x)$, as a result $d(T^{n+N} x, T^{n+N} y) \leq 2^{-(n+1)}$ for every $n \geq 0$. Since $2^{-(n+1)}$ tends to zero as n tends to infinity we get $\lim_{n\to\infty} d(T^n x, T^n y) = 0$. Reciprocally, if $\lim_{n\to\infty} d(T^n x, T^n y) = 0$ there exists $N > 0$ such that $d(T^n x, T^n y) < 1/2$ for every $n \geq N$. In particular, $(T^n x)_0 = (T^n y)_0$ for every $n \geq N$. Hence $T^N y \in W^s_{loc}(T^N x)$ and $y \in T^{-N}(W^s_{loc}(T^N x)) \subset W^s(x)$. We get $W^u(x) = \{y \in X : \lim_{n\to\infty} d(T^{-n} x, T^{-n} y) = 0\}$ analogously. ■

2.2 Anosov diffeomorphisms

We recall some basic definitions and properties of Anosov diffeomorphisms which are going to be useful throughout the proof of Theorem 1.5. During this section we use [BS], [KH] and [Sh] as a general reference.

Definition 2.6. Let X be a connected smooth manifold. A diffeomorphism $T : X \to X$ is called an *Anosov diffeomorphism* if there is an invariant decomposition of the tangent bundle TX as a direct sum of continuous DT-invariant sub-bundles E_x^s and E_x^u such that, for some appropriate Riemannian metric,

$$\|DT_x^n(v^s)\| \leq c\lambda^n \|v^s\| \quad \text{and} \quad \|(DT_x^n)^{-1}(v^u)\| \leq c\lambda^n \|v^u\|,$$

for all $x \in X$ and for any pair of unit vectors $v^s \in E_x^s$, $v^u \in E_x^u$, where $0 < \lambda < 1$ and $c > 0$ are both constants.

We proceed to show a basic proposition about the dependence of E_x^s and E_x^u on x.

Proposition 2.7. *Let $T : X \to X$ be an Anosov diffeomorphism. Then the subspaces E_x^s and E_x^u depend continuously on x.*

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of points in X converging to $x \in X$. By passing to a subsequence, we may assume that $k := \dim(E_{x_i}^s)$ is constant. Let $w_{1,i}, \ldots, w_{k,i}$ be an orthonormal basis in $E_{x_i}^s$. By the compactness of T^1X there is a subsequence $(w_{j,i})_{i\in\mathbb{N}}$ which converges to $w_j \in T^1X$ for each $j \in \{1, \ldots, k\}$. Since $\|DT_x(v^s)\| \leq c\lambda\|v^s\|$ is a closed condition, each vector from the orthonormal frame $w_1, \ldots, w_k$ satisfies it. Moreover, by the invariance of E_x^s we get that w_j lies in E_x^s for each $j \in \{1, \ldots, k\}$. It follows that $\dim(E_x^s) \geq k = \dim(E_{x_i}^s)$. A similar argument shows that $\dim(E_x^u) \geq \dim(E_{x_i}^u) = \dim(X) - k$. Hence, $\dim(E_x^s) = \dim(E_{x_i}^s)$ and $\dim(E_x^u) = \dim(E_{x_i}^u)$ for every $i \in \mathbb{N}$. So the subspaces E_x^s and E_x^u depend continuously on x. ■

The notion of an Anosov diffeomorphism does not depend on the choice of the Riemannian metric on X, however the constant C depends on the metric. As the next proposition shows, we can change the metric to get $C = 1$ by using a larger λ.

Proposition 2.8. *Let $T : X \to X$ be an Anosov diffeomorphism with constants C and λ. Then for every $\epsilon > 0$ there is a C^1-Riemannian metric $(\cdot,\cdot)'$ on X called* adapted metric, *with respect to which T is an Anosov diffeomorphism with constants $C' = 1$ and $\lambda' = \lambda + \epsilon$ and $(v^s, v^u)' < \epsilon$ for every $x \in X$ and every pair of unit vectors $v^s \in E_x^s$, $v^u \in E_x^u$.*

Proof. Let $x \in X$ and a pair of vectors $v^s \in E_x^s$, $v^u \in E_x^u$. We define

$$\|v^s\|' := \sum_{n=0}^{\infty}(\lambda + \epsilon)^{-n}\|DT_x^n v^s\| \quad \text{and} \quad \|v^u\|' := \sum_{n=0}^{\infty}(\lambda + \epsilon)^{-n}\|DT_x^{-n} v^u\|.$$

Both series converge uniformly for every $x \in X$ and every pair of vectors $v^s \in E^s_x$, $v^u \in E^u_x$. So

$$\|DT_x v^s\|' = \sum_{n=0}^{\infty} (\lambda+\epsilon)^{-n} \|DT_x^{n+1} v^s\| = (\lambda+\epsilon)(\|v^s\|' - \|v^s\|) \leq (\lambda+\epsilon)\|v^s\|',$$

and similarly for $\|DT_x^{-1} v^u\|'$. For an arbitrary vector $v = v^s + v^u \in T_x X$, we define

$$\|v\|' := \sqrt{(\|v^s\|')^2 + (\|v^u\|')^2}.$$

Moreover, the metric is recovered from the norm by defining

$$(v, w)' := \frac{\|v+w\|'^2 - \|v\|'^2 - \|w\|'^2}{2}.$$

Respect to this metric E^s_x and E^u_x are orthogonal and T is an Anosov diffeomorphism with constants $c = 1$ and $\lambda + \epsilon$ instead of λ. Finally, by standard methods of differential topology $(\cdot, \cdot)'$ can be uniformly approximated by a smooth metric defined on X. See [H] for more details. ■

Let us recall the following fundamental result about stable and unstable manifolds for an Anosov diffeomorphism. Let d be the Riemannian distance function.

Theorem 2.9. *(Stable Manifold Theorem) Let $T : X \to X$ be an Anosov diffeomorphism of class C^k. Then there exist $\epsilon_0 > 0$ and $0 < \lambda < 1$ such that for each $0 < \epsilon < \epsilon_0$ and $x \in X$, the local stable manifold*

$$W^s_{loc}(x) := \{y \in X : d(T^n x, T^n y) \leq \epsilon \quad \textit{for all } n \geq 0\},$$

and the local unstable manifold

$$W^u_{loc}(x) := \{y \in X : d(T^{-n} x, T^{-n} y) \leq \epsilon \quad \textit{for all } n \geq 0\},$$

are C^k embedded disks tangent at x to E^s_x and E^u_x respectively. In addition,

- *$T(W^s_{loc}(x)) \subset W^s_{loc}(Tx)$ and $T^{-1}(W^u_{loc}(x)) \subset W^u_{loc}(T^{-1}x)$;*
- *$d(T(x), T(y)) \leq \lambda d(x, y)$ for all $y \in W^s_{loc}(x)$;*
- *$d(T^{-1}(x), T^{-1}(y)) \leq \lambda d(x, y)$ for all $y \in W^u_{loc}(x)$;*
- *$W^s_{loc}(x)$ and $W^s_{loc}(x)$ vary continuously with the point x in the C^k topology.*

Furthermore, the global stable and unstable manifolds of x,

$$W^s(x) := \bigcup_{n=0}^{\infty} T^{-n}(W^s_{loc}(T^n x)) \quad \textit{and} \quad W^u(x) := \bigcup_{n=0}^{\infty} T^{n}(W^u_{loc}(T^{-n} x)),$$

are smoothly immersed submanifolds of X.

Before to prove the Stable Manifold Theorem we are going to prove the Hadamard-Perron's Theorem. In the following δ is a positive constant and $B_\delta \subset \mathbb{R}^d$ denotes the ball of radius δ centered at 0.

Theorem 2.10. *(Hadamard-Perron's Theorem) Let $f := (f_n)_{n\in\mathbb{N}}$ be a sequence of C^1 diffeomorphisms $f_n : B_\delta \to \mathbb{R}^d$ onto their images such that $f_n(0) = 0$ for every positive integer n. Let us suppose the existence of a constant $\lambda \in (0,1)$ and a splitting $\mathbb{R}^d = E^s(n) \oplus E^u(n)$ for each positive integer n such that*

1. *$Df_n(0)E^s(n) = E^s(n+1)$ and $Df_n(0)E^u(n) = E^u(n+1)$.*
2. *$\|Df_n(0)v^s\| \le \lambda \|v^s\|$ for every $v^s \in E^s(n)$.*
3. *$\|Df_n(0)^{-1}v^u\| \le \lambda \|v^u\|$ for every $v^u \in E^u(n)$.*
4. *there is a positive constant $c > 0$ such that $\measuredangle(E^s(n), E^u(n)) \ge c$ for every positive integer n.*
5. *$\{Df_n(0)(\cdot)\}_{n\in\mathbb{N}}$ is an equicontinuous family of functions $Df_n(0)(\cdot) : B_\delta \to GL(d,\mathbb{R})$.*

Then, there is a positive constant $\epsilon > 0$ and a sequence $\phi := \{\phi_n\}_{n\in\mathbb{N}}$ of Lipschitz maps $\phi_n : B^s_\epsilon := \{v \in E^s(n) : \|v\| < \epsilon\} \to E^s(n)$ such that

1. *$graph(\phi_n) \bigcap B^s_\epsilon = W^s_\epsilon(n) := \{x \in B_\epsilon : \lim_{k\to\infty} \|f_{n+k-1}(\cdots(f_{n+1}(f_n(x)))\cdots)\| = 0$.*
2. *$f_n(graph(\phi_n)) \subset graph(\phi_{n+1})$ for every positive integer n.*
3. *$\|f_n(x)\| \le \lambda \|x\|$ for every $x \in graph(\phi_n)$.*
4. *For every $x \in B_\epsilon \setminus graph(\phi_n)$*

   $$\|P^u_n x - \phi_n(P^s_n x)\| \le \lambda \|P^u_{n+1} f_n(x) - \phi_{n+1}(P^s_{n+1} f_n(x))\|,$$

   where P^s_n and P^u_n denote the projection on $E^s(n)$ and $E^u(n)$ respectively.
5. *ϕ_n is differentiable at 0 and $D\phi_n(0) = 0$ for every positive integer n.*

6. *ϕ depends continuously on f in the topologies induced by*

$$d(\phi, \psi) = \sup_{n \in \mathbb{N}, x \in B_\epsilon} |\phi_n(x) - \psi_n(x)|$$

$$d_2(f, g) = \sup_{n \in \mathbb{N}} 2^{-n} d_1(f_n, g_n)$$

where d_1 denotes the C^1 metric.

Proof. Let L and ϵ two arbitrary positive constants. We define the space $\Phi(L, \epsilon)$ of sequences $\phi = (\phi_n)_{n \in \mathbb{N}}$ where $\phi_n : B^s_\epsilon \to E^u(n)$ is a Lipschitz mapping of constant L such that $\phi_n(0) = 0$ for every integer n. Note that $\Phi(L, \epsilon)$ is a complete metric space with the metric defined by

$$d(\phi, \psi) = \sup_{n \in \mathbb{N}, x \in B_\epsilon} |\phi_n(x) - \psi_n(x)|.$$

We define the operator $F : \Phi(L, \epsilon) \to \Phi(L, \epsilon)$ by $F(\phi) = \psi = (\psi_n)_{n \in \mathbb{N}}$ where $\psi_n : B^s_\epsilon \to E^u_\epsilon(n)$ is the unique Lipschitz mapping of constant L which has its graph contained in $f_n^{-1}(graph(\phi_{n+1}))$. This operator is called *the graph transform*.

Note that a map $h : \mathbb{R}^k \to \mathbb{R}^l$ is Lipschitz continuous at $0 \in \mathbb{R}^k$ with Lipschitz constant L if and only if the graph of h lies in the L-cone K about $\mathbb{R}^k$. More generally, it is Lipschitz continuous at $x \in \mathbb{R}^k$ if and only if its graph lies in the L-cone K about the translate by $(x, h(x))$.

Lemma 2.11. *For any $L > 0$ there exists $\epsilon > 0$ such that the graph transform $F : \Phi(L, \epsilon) \to \Phi(L, \epsilon)$ is a well-defined operator.*

Proof. Let $L > 0$ and $x \in B_\epsilon$. Let us define the stable cone $K^s_L(n)$ as the set

$$K^s_L(n) := \{v \in \mathbb{R}^m : v = v^s + v^u, v^s \in E^s(n), v^u \in E^u(n), \|v^u\| \leq L\|v^s\|\}.$$

By definition $Df_n^{-1}(0)K^s_L(n+1) \subset K^s_L(n)$. By the uniform continuity of Df_n, for any $L > 0$ there exists $\epsilon > 0$ such that $Df_n^{-1}(x)K^s_L(n+1) \subset K^s_L(n)$ for every $n \in \mathbb{N}$ and $x \in B_\epsilon$. Hence, the preimage under f_n of the graph of a Lipschitz function is the graph of a Lipschitz function. For $\phi \in \Phi(L, \epsilon)$, we consider the following composition $\beta := P^s(n) \circ f_n^{-1} \circ \phi_n$, where $P^s(n)$ is the projection onto $E^s(n)$ parallel to $E^u(n)$. If ϵ is small enough, then β is an expanding map and its image covers $B^s(n)$. Hence $F(\phi) \in \Phi(L, \epsilon)$ and the graph transform F is a well-defined. ■

Lemma 2.12. *There are two positive constants $L > 0$ and $\epsilon > 0$ such that the graph transform F is a contraction on $\Phi(L, \epsilon)$.*

Proof. Let us define the unstable cone $K_L^u(n)$ as the set

$$K_L^u(n) := \{v \in \mathbb{R}^m : v = v^s + v^u, v^s \in E^s(n), v^u \in E^u(n), L\|v^u\| \geq \|v^s\|\}.$$

By definition $Df_n(0)K_L^u(n) \subset K_L^u(n+1)$. As in Lemma 2.11, by the uniform continuity of Df_n, for any $L > 0$ there exists $\epsilon > 0$ such that the inclusion $Df_n K_L^u(n) \subset K_L^u(n+1)$ holds for every integer $n > 0$ and $x \in B_\epsilon$. Let us consider $\phi, \psi \in \Phi(L, \epsilon)$, $\tilde{\phi} = F(\phi)$ and $\tilde{\psi} = F(\psi)$. For any $\eta > 0$ there exist $n \in \mathbb{N}$ and $y \in B^s$ such that $|\tilde{\phi}(y)_n - \tilde{\psi}_n(y)| > d(\tilde{\phi}, \tilde{\psi}) - \eta$. Let c^u be the straight line segment from $(y, \tilde{\phi}_n(y))$ to $(y, \tilde{\psi}_n(y))$. Since c^u is parallel to $E^u(n)$, then $length(f_n(c^u)) > \lambda^{-1} length(c^u)$. Let $f_n(y, \tilde{\psi}_n(y)) = (z, \tilde{\psi}_{n+1}(z))$ and consider the curvilinear triangle formed by the straight line segment from $(z, \phi_{n+1}(z))$ to $(z, \psi_{n+1}(z))$, $f_n(c^u)$, and the shortest curve on the graph of $\tilde{\phi}_{n+1}$ connecting the ends of these curves. For small enough $\epsilon > 0$ the tangent vectors to the image $f_n(c^u)$ lie in $K_L^u(n+1)$ and the tangent vectors to the graph of ϕ_{n+1} lie in $K_L^s(n+1)$. Therefore,

$$\begin{aligned}
|\phi_{n+1}(z) - \psi_{n+1}(z)| &\geq \frac{length(f_n(c^u))}{1+2L} - L(1+L) \cdot length(f_n(c^u)) \\
&\geq (1-4L) length(f_n(c^u))
\end{aligned}$$

and

$$\begin{aligned}
d(\phi, \psi) &\geq |\phi_{n+1}(z) - \psi_{n+1}(z)| \geq (1-4L) length(f_n(c^u)) \\
&\geq (1-4L)\lambda^{-1} length(c^u) = (1-4L)\lambda^{-1}(d(\tilde{\phi}, \tilde{\psi}) - \eta).
\end{aligned}$$

Since η is arbitrary F is contracting for small enough L and ϵ. ■

Since F is contracting and depends continuously on f, it has a unique fixed point $\phi \in \Phi(L, \epsilon)$ which depends continuously on f and automatically satisfies property 2. For small enough ϵ, the invariance of the stable and unstable cones (with a small enough ϵ) implies that ϕ satisfies properties 3 and 4. Moreover, property 1 follows immediately from 3 and 4. Since property 1 gives a geometric characterization of $graph(\phi_n)$, the fixed point of F for a smaller ϵ is a restriction of the fixed point of F for a larger ϵ to a smaller domain. Furthermore, as $\epsilon \to 0$ and $L \to \infty$ the stable cone $K_L^s(n)$ tends to $E^s(n)$. Therefore $E^s(n)$ is the tangent plane to $graph(\phi_n)$ at 0. ■

Now, we are able to prove the Stable Manifold Theorem.

Proof. By the compactness of X, there is a finite collection U of charts $\{(U_{x_j}, \psi_{x_j})\}_{1\leq j\leq m}$ such that U_{x_j} covers the ball $B(x_j, \delta)$ centered at x_j of radius δ for some small positive constant δ. Moreover, the changes of coordinates $\psi_{x_i} \circ \psi_{x_j}^{-1}$ have equicontinuous first derivatives. For any $x \in X$, we define $T_n = \psi_{T^n x} \circ T \circ \psi_{T^{n-1}x}^{-1}$, $E^s(n) = D_{T^n x}\psi(x)E^s(T^n x)$ and $E^u(n) = D_{T^n x}\psi(x)E^u(T^n x)$. By the Hadamard-Perron's Theorem we get the local stable manifolds $W^u_\epsilon(x)$. Analogously, applying the Hadamard-Perron's Theorem to T^{-1} we get the local unstable manifolds $W^u_\epsilon(x)$. All the properties of the local stable manifolds $W^s_\epsilon(x)$ and the local unstable manifolds $W^u_\epsilon(x)$ follow directly from the Hadamard-Perron's Theorem. ∎

Proposition 2.13. *The global stable and unstable manifolds of x are characterized by*

$$W^s(x) = \{y \in X : \lim_{n\to\infty} d(T^n x, T^n y) = 0\} \quad \textit{and} \quad W^u(x) = \{y \in X : \lim_{n\to\infty} d(T^{-n}x, T^{-n}y) = 0\}.$$

Proof. Let $y \in W^s(x) = \bigcup_{n=0}^{\infty} T^{-n}(W^s_{loc}(T^n x))$ by definition there exists N such that $T^N y \in W^s_{loc}(T^N x)$, as a result $d(T^{n+N}x, T^{n+N}y) \leq \epsilon$ for every $n \geq 0$. Since $\epsilon < \epsilon_0$ can be chosen arbitrarily small we get that $\lim_{n\to\infty} d(T^n x, T^n y) = 0$. Reciprocally, if $\lim_{n\to\infty} d(T^n x, T^n y) = 0$ there exists $N > 0$ such that $d(T^n x, T^n y) < \epsilon$ for every $n \geq N$. In particular, $T^N y \in W^s_{loc}(T^N x)$ and $y \in T^{-N}(W^s_{loc}(T^N x)) \subset W^s(x)$. We get $W^u(x) = \{y \in X : \lim_{n\to\infty} d(T^{-n}x, T^{-n}y) = 0\}$ analogously. ∎

Example. Let us consider a diffeomorphism $T : \mathbb{T}^2 \to \mathbb{T}^2$ induced by the matrix

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since the one-dimensional subspaces generated by $\Big(\frac{1+\sqrt{5}}{2}, 1\Big)$ and $\Big(\frac{1-\sqrt{5}}{2}, 1\Big)$ are eigenspaces of A, for each point $x \in \mathbb{T}^2$ we have a constant A-invariant decomposition

$$E^u_x \oplus E^s_x = span\Big(\Big(\frac{1+\sqrt{5}}{2}, 1\Big)\Big) \oplus span\Big(\Big(\frac{1-\sqrt{5}}{2}, 1\Big)\Big)$$

of $T_x\mathbb{T}^2$. Obviously E^u_x and E^s_x are continuous sub-bundles as they are constant. Finally,

$$\|A(v^s)\| = \Big(\frac{3+\sqrt{5}}{2}\Big)^{-1} < 1 < \frac{3+\sqrt{5}}{2} = \|A(v^u)\|,$$

for all $x \in X$ and for any pair of unit vectors $v^s \in E^s_x$, $v^u \in E^u_x$. In other words, by defining $c = 1$ and $\lambda = \Big(\frac{3+\sqrt{5}}{2}\Big)^{-1}$ the diffeomorphism T is Anosov. Note that $\frac{3+\sqrt{5}}{2}$ corresponds to the greater eigenvalue of the matrix A.

More generally, each matrix $A \in SL(d, \mathbb{Z})$ induces an Anosov diffeomorphism $T : \mathbb{T}^d \to \mathbb{T}^d$. These Anosov diffeomorphisms are called *linear Anosov diffeomorphisms*.

Another well-known property of Anosov dynamics is their *local product structure*. More precisely, there is a constant $\delta_1 > 0$ such that for every $x, y \in X$ which satisfy $d(x, y) < \delta_1$ the intersection $W^u_{loc}(x) \bigcap W^s_{loc}(y)$ consists of a unique point denoted by $[x, y]$. In fact, for ϵ small enough the local stable manifold $W^s_\epsilon(x)$ and the local unstable manifold $W^u_\epsilon(x)$ have transversal intersection at x and these manifolds vary C^1-continuously respect to x. As a result, we get the local product structure.

2.2.1 Closing property

During this subsection we prove a well-known property satisfy by Anosov diffeomorphisms. This property will be crucial in the proof of Theorem 1.5.

Definition 2.14. A sequence $x_0, x_1, \ldots, x_n = x_0$ of points is called a *periodic ϵ-pseudo-orbit* if $d(T(x_k), x_{k+1}) < \epsilon$ for $k = 0, 1, \ldots, n-1$.

Definition 2.15. An homeomorphism $T : X \to X$ satisfies the *closing property* if there exist two positive constants C, δ_0 such that for $\epsilon < \delta_0$ any periodic ϵ-pseudo-orbit $x_0, x_1, \ldots, x_n = x_0$, there is a periodic point p such that $T^n p = p$ and $d(T^k p, x_k) < C\epsilon$, for every $k \in \{0, 1, \ldots, n\}$.

Remark. In particular, if a homeomorphism T satisfies the closing property and $x \in X$ satisfies $d(x, T^n x) < \delta_0$, then there is a periodic point $p = T^n p$ such that $d(T^k x, T^k p) < C\epsilon$ for every $k \in \{0, 1, \ldots, n\}$.

In the following, we show that Anosov diffeomorphisms satisfy this property. We follow the proof given by Katok and Hasselblatt in [KH].

Proposition 2.16. *(Anosov closing lemma) Every Anosov diffeomorphism $T : X \to X$ satisfies the closing property.*

Proof. For every $x_k \in X$ there is a neighborhood V_k on which T is a small perturbation of a hyperbolic linear map given by

$$T_k(u, v) = (A_k u + \alpha_k(u, v), B_k v + \beta_k(u, v)),$$

where $\|\alpha_k\|$, $\|\beta_k\|$, $\|D\alpha_k\|$ and $\|D\beta_k\|$ are bounded by $C\epsilon$ for all $k \in \{0, \ldots, n-1\}$ for some positive constant C. We recall that each linear map corresponds to the derivative at the respective point and we do not assume the maps T_k fix the origin. A sequence $((u_k, v_k))_{k\in 0,\ldots,n-1}$ of elements in $(u_k, v_k) \in V_k$ is a periodic orbit if and only if

$$(u, v) := ((u_0, v_0), (u_1, v_1), \ldots, (u_{n-1}, v_{n-1}))$$

$$= (T_{n-1}(u_{n-1}, v_{n-1}), T_0(u_0, v_0), \ldots, T_{n-2}(u_{n-2}, v_{n-2})) =: F(u, v).$$

Therefore, we need to find a fixed point of the map $F : \mathbb{R}^n \to \mathbb{R}^n$. In the following, we use the norm $\|(x_0, x_1, \ldots, x_{n-1})\| := \max_{0\leq i\leq n-1} \|x_i\|$ on $\mathbb{R}^n$. Let us represent F as a perturbation of a linear map L as $F(u, v) = L(u, v) + S(u, v)$, where

$$S((u_0, v_0), (u_1, v_1), \ldots, (u_{n-1}, v_{n-1}))$$

$$:= ((\alpha_{n-1}(u_{n-1}, v_{n-1}), \beta_{n-1}(u_{n-1}, v_{n-1})), \ldots, (\alpha_{n-2}(u_{n-2}, v_{n-2}), \beta_{n-2}(u_{n-2}, v_{n-2}))),$$

$$L((u_0, v_0), (u_1, v_1), \ldots, (u_{n-1}, v_{n-1}))$$

$$:= ((A_{n-1}(u_{n-1}), B_{n-1}(v_{n-1})), (A_0 u_0, B_0 v_0), \ldots, (A_{n-2}(u_{n-2}), B_{n-2}(v_{n-2}))).$$

Note the linear mapping L is hyperbolic because it expands the subspace generated by the vectors $(u_0, 0), (u_1, 0), \ldots, (u_{n-1}, 0)$ and contracts the subspace generated by the vectors $(0, v_0), (0, v_1), \ldots, (0, v_{n-1})$. Since $\|S(u, v) - S(u', v')\| \leq \widetilde{C}\epsilon\|(u, v) - (u', v')\|$ for some positive constant $\widetilde{C} = \widetilde{C}(T, \Lambda)$, we can apply the Hyperbolic Fixed-Point theorem to obtain a periodic point $p = T^n p$ such that $d(T^k p, x_k) < C\epsilon$ for every $k \in \{0, 1, \ldots, n\}$. ■

Furthermore, it is trivial to check that subshifts of finite type also satisfy this property. For more details see [KH].

We say a point $x \in X$ is a *non-wandering* point if for any neighbourhood U of x there exists $n \in \mathbb{N}$ such that $T^n U \bigcap U \neq \emptyset$. Let $NW(T)$ be the set of non-wandering points of T. Note that $NW(T)$ is T-invariant and closed.

Corollary 2.17. *Let $T : X \to X$ an Anosov diffeomorphism. Then the set of periodic points is dense in the set $NW(T)$ of non-wandering points of T.*

Proof. Let $x \in NW(T)$ and $\epsilon > 0$. Set $U_\epsilon := B(x, \epsilon/2)$ the ball of radius $\epsilon/2$ centered on x. There exists a positive integer N such that $T^N(U_\epsilon) \bigcap U_\epsilon \neq \emptyset$. In particular, for $y \in T^N(U_\epsilon) \bigcap U_\epsilon$ there exists $z \in U_\epsilon$ such that $y = T^N z \in U_\epsilon$ and $d(z, T^N z) < \epsilon$. If $\epsilon < \delta_0$, by the closing lemma there is a periodic point p such that $d(T^k p, T^k z) < C\epsilon$ for $k \in \{1, 2, \ldots, N-1\}$. Finally, note that $d(x, p) \leq d(x, z) + d(z, p) \leq \dfrac{\epsilon}{2} + C\epsilon = \epsilon\Big(\dfrac{1}{2} + C\Big)$. Hence the set of periodic points is dense in $NW(T)$. ■

Remark. Note that every T-invariant probability measure μ is supported on $NW(T)$. Otherwise, there is a T-invariant probability measure $\tilde{\mu}$ such that $\mu(X \setminus NW(T)) > 0$. Let $x \in (X \setminus NW(T)) \bigcap supp(\tilde{\mu})$, by definition there exists a open neighbourhood U of x such that $\tilde{\mu}(U) > 0$ and $U \bigcap T^n U = \emptyset$ for every $n > 0$. Hence, we obtain a family of disjoint sets $\{T^n U\}_{n>0}$ which have the same non-zero measure. This gives a contradiction since $\tilde{\mu}$ is a probability measure.

Chapter 3

Linear Cocycles

3.1 Linear cocycles

A linear cocycle is a pair (T, A) where $T : X \to X$ is a homeomorphism defined on a compact metric space X and $A : X \to GL(d, \mathbb{R})$ is a continuous function. In particular, a $SL(2, \mathbb{R})$-cocycle is a linear cocycle where A takes values on $SL(2, \mathbb{R})$. Besides, we are going to use the following notation

$$A^n(x) := A(T^{n-1}x)A(T^{n-2}x)\dots A(Tx)A(x),$$

$$A^{-n}(x) := A(T^{-n}x)^{-1}A(T^{-n+1}x)^{-1}\dots A(T^{-2}x)^{-1}A(T^{-1}x)^{-1},$$

and $A^0(x) := I$ for every $x \in X$ and $n > 0$. In particular,

$$A^{n+m}(x) = A(T^{n+m-1}x)\cdots A(x) = A^n(T^m x)A^m(x)$$

for every $n, m \in \mathbb{N}$ and $x \in X$. More precisely, we can think a linear cocycle (T, A) as a mapping $F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ defined by $F(x, v) = (Tx, A(x)v)$. We are interested in the dynamic defined by F. In fact, the previous definition of $A^n(x)$ is motivated by the identity $F^n(x, v) = (T^n x, A^n(x)v)$.

The most basic example of a linear cocycle are the one-step cocycles.

Example. Let $X = \{1, 2, \dots, n\}^{\mathbb{Z}}$ be the space of bi-infinite sequences on n symbols. Let $T : X \to X$ the left-shift map. Given a set of matrices $\{A_1, A_2, \dots, A_n\} \subset GL(d, \mathbb{R})$ we define the function $A : X \to GL(d, \mathbb{R})$ by $A(x) = A_{x_0}$. In this case, we say that (T, A) is a *one-step cocycle*.

We proceed to show another well-known and big class of cocycles. These cocycles have a nice geometric interpretation of the function A as the derivative of a diffeomorphism T.

Example. Let $T : X \to X$ be a diffeomorphism defined on a smooth Riemannian manifold X of dimension d. If the manifold X is parallelizable, we can construct a family of smooth vector fields $e_1, \ldots, e_d$ such that $e_1(x), \ldots, e_d(x)$ is a basis of the tangent space T_xX for every $x \in X$. We define *the derivative cocycle over the diffeomorphism* T by the function $A : X \to GL(d, \mathbb{R})$ where $A(x)$ is the matrix that represents the linear mapping $DT_x : T_xX \to T_{T(x)}X$ respect to the basis defined by $e_1, \ldots, e_d$.

Remark. By the chain's rule we have

$$A^n(x) = A(T^{n-1}x) \cdots A(x) = DT_{T^{n-1}x} \cdots DT_x = D_x(T^n)$$

in the previous example. Hence, it would be interesting to get some information about $A^n(x) = D_x(T^n)$. In the following section we will prove the theorem of Furstenberg and Kesten which says that the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| = \lim_{n\to\infty} \frac{1}{n} \log \|D_x(T^n)\|$$

is well defined in a large set of points $x \in X$.

3.1.1 Uniformly hyperbolic cocycles

As we mentioned in the introduction we are going to prove a characterization of uniform hyperbolicity. For completeness, we proceed to present its definition once again.

Definition 3.1. A $SL(2, \mathbb{R})$-cocycle (T, A) is called *uniformly hyperbolic* if there are constants $c > 0$ and $0 < \lambda < 1$ such that for every $x \in X$ there exist transverse one-dimensional spaces E^s_x and E^u_x in $\mathbb{R}^2$ such that

1. $A(x)E^s_x = E^s_{T(x)}$ and $A(x)E^u_x = E^u_{T(x)}$,
2. $\|A^n(x)v^s\| \leq c\lambda^n\|v^s\|$ and $\|A^{-n}(x)v^u\| \leq c\lambda^n\|v^u\|$,

for every $x \in X$, $v^s \in E^s_x$, $v^u \in E^u_x$ and $n \geq 1$.

Here $\|\cdot\|$ denotes the Euclidean norm.

Remark. The subspaces E^u_x and E^s_x that satisfy the properties above are unique. If for some x there exists two linearly independent unit vectors $v_1, v_2 \in \mathbb{R}^2$ such that

$\|A^n(x)v_i\| \le c\lambda^n\|v_i\|$ then $\lim_{n\to\infty} A^n(x)v_i = 0$ for both $i = 1, 2$. Then we would have $\lim_{n\to\infty} \|A^n(x)\| = 0$, which is impossible since

$$1 = \|A^n(x) \cdot A^n(x)^{-1}\| \le \|A^n(x)\| \cdot \|A^n(x)^{-1}\| = \|A^n(x)\|^2, \quad \text{for every } x \in X.$$

For $SL(2,\mathbb{R})$-cocycles there is a well known characterization of uniform hyperbolicity proved by J.-C. Yoccoz in [Y] (see [BG] and [Z] for related results).

Proposition 3.2. *A $SL(2,\mathbb{R})$-cocycle (T, A) is uniformly hyperbolic if and only if there are constants $c > 0$ and $\tau > 0$ such that*

$$\|A^n(x)\| \ge ce^{\tau n}, \quad \textit{for all } n \ge 0 \textit{ and } x \in X.$$

Proof. Let us suppose that (T, A) is uniformly hyperbolic. By definition

$$\|A^n(x)\| = \sup_{v\neq 0} \frac{\|A^n(x)v\|}{\|v\|} \ge \frac{\|A^n(x)v^u\|}{\|v^u\|} \ge c^{-1}\lambda^{-n} \quad \text{for every } x \in X \text{ and } n \in \mathbb{N}.$$

So we get the result automatically. Now, we suppose that $\|A^n(x)\| \ge ce^{\tau n}$ for all $n \ge 0$ and $x \in X$. Since $e^\tau > 1$, then$\|A^n(x)\| > 1$ for every large enough n. Let $u_n(x)$ the most expanded unit vector by $A^n(x)$ and $s_n(x)$ the most contracted unit vector by $A^n(x)$ which both exist and are unique by the following basic linear algebra lemma.

Lemma 3.3. *Let $M \in SL(2,\mathbb{R})$ such that $\|M\| \neq 1$. There exist unit vectors u and s such that $\|Mu\| = \|M\|$ and $\|Ms\| = \|M^{-1}\|^{-1}$. These vectors are unique up to multiplication by -1. Moreover the vectors u and s are orthogonal and their images Ms and Mu are orthogonal.*

By Lemma 3.3 the vectors $u_n(x)$ and $s_n(x)$ are characterized by

$$\|A^n(x)u_n(x)\| = \|A^n(x)\| \quad \text{and} \quad \|A^n(x)s_n(x)\| = \|A^n(x)^{-1}\|^{-1} = \|A^n(x)\|^{-1}.$$

We proceed to prove that $\{s_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. We will see the limit $s(x)$ of $\{s_n(x)\}_{n\in\mathbb{N}}$ define the one-dimensional subspace E^s_x in the definition of uniform hyperbolicity.

Lemma 3.4. *There are positive constants C_1, C_2 such that*

$$|\sin \measuredangle(s_n(x), s_{n+1}(x))| \le C_1 e^{-n\tau}\|A^n(x)\|^{-1} \le C_2 e^{-2n\tau},$$

for every integer $n > 0$ and every point $x \in X$.

Proof. Let $\alpha_n := \measuredangle(s_n(x), s_{n+1}(x))$. Since $u_{n+1}(x)$ and $s_{n+1}(x)$ form an orthonormal basis then $s_n(x) = \sin(\alpha_n)u_{n+1}(x) + \cos(\alpha_n)s_{n+1}(x)$. Moreover $A^{n+1}(x)u_{n+1}(x)$ and

$A^{n+1}(x)s_{n+1}(x)$ are orthogonal, so

$$\|A^{n+1}(x)s_n(x)\| \geq \|\sin(\alpha_n)\cdot A^{n+1}(x)u_{n+1}(x)\| = |\sin(\alpha_n)|\cdot\|A^{n+1}(x)\|.$$

Furthermore, by the submultiplicativity of the norm

$$\|A^{n+1}(x)s_n(x)\| \leq \|A(T^n x)\|\cdot\|A^n(x)s_n(x)\| = \|A(T^n x)\|\cdot\|A^n(x)\|^{-1}.$$

By the previous two inequalities,

$$\begin{aligned} |\sin(\alpha_n)| \leq & \|A(T^n x)\|\cdot\|A^{n+1}(x)\|^{-1}\cdot\|A^n(x)\|^{-1} \\ \leq & Cc^{-1}e^{-\tau}e^{-n\tau}\|A^n(x)\|^{-1} \leq Cc^{-2}e^{-\tau}e^{-2n\tau}, \end{aligned}$$

where C is an upper bound for $\|A(x)\|$ which exists since X is compact. This proves the lemma by considering $C_1 := Cc^{-1}e^{-\tau}$ and $C_2 = Cc^{-2}e^{-\tau}$. ■

From Lemma 3.4 we get

$$\limsup_{n\to\infty}\frac{1}{n}\log|\sin(\alpha_n)| \leq \lim_{n\to\infty}\left(\frac{1}{n}\log C_2 + \frac{1}{n}\log e^{-2n\tau}\right) = -2\tau,$$

so $|\sin(\alpha_n)| \leq e^{n(-2\tau+\epsilon)}$ for every n large enough. Up to replacing s_n by $-s_n$ we can assume that $\sin(\alpha_n)\geq 0$ for every n large enough. So

$$\begin{aligned} \|s_{n+1}(x)-s_n(x)\|^2 &= (s_{n+1}(x)-s_n(x), s_{n+1}(x)-s_n(x)) \\ &= 2(1-\cos(\alpha_n)) \leq 2(1-\sqrt{1-e^{2n(-2\tau+\epsilon)}}) \\ &\leq 2e^{2n(-2\tau+\epsilon)}. \end{aligned}$$

More generally, by last inequality we get

$$\begin{aligned} \|s_{n+k}(x)-s_n(x)\| &\leq \|s_{n+k}(x)-s_{n+k-1}(x)\| + \cdots + \|s_{n+1}(x)-s_n(x)\| \\ &\leq \sqrt{2}e^{(n+k)(-2\tau+\epsilon)} + \cdots + \sqrt{2}e^{n(-2\tau+\epsilon)} \\ &\leq \sqrt{2}e^{n(-2\tau+\epsilon)}(e^{(-2\tau+\epsilon)k} + \cdots + 1) \\ &\leq C_2 e^{n(-2\tau+\epsilon)}, \end{aligned}$$

where $C_2 = \sqrt{2}\sum_{k=0}^{\infty} e^{(-2\tau+\epsilon)k}$ for ϵ small enough. Hence, $\{s_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of unit vectors, so $s(x) := \lim_{n\to\infty} s_n(x)$ is well-defined. Moreover, by Lemma 3.4

$$|\sin\measuredangle(s_{n+k}(x), s_n(x))| \leq \sum_{m=n}^{n+k-1}|\sin\measuredangle(s_{m+1}(x), s_m(x))| \leq C_1\sum_{m=n}^{n+k-1} e^{-m\tau}\|A^m(x)\|^{-1}.$$

Consequently

$$|\sin\measuredangle(s(x), s_n(x))| \leq C_1 \sum_{m=n}^{\infty} e^{-m\tau}\|A^m(x)\|^{-1} \leq C_2 \sum_{m=n}^{\infty} e^{-2m\tau}.$$

Lemma 3.5. *The vectors $A(x)s(x)$ and $s(T(x))$ are collinear for every $x \in X$.*

Proof. Let $\beta_n := \measuredangle(A(x)s_{n+1}(x), s_n(T(x)))$. Since $u_n(T(x))$ and $s_n(T(x))$ form an orthonormal basis $A(x)s_{n+1}(x) = \sin(\beta_n)u_n(T(x)) + \cos(\beta_n)s_n(T(x))$ and

$$\|A^{n+1}(x)s_{n+1}(x)\| \geq |\sin\beta_n| \cdot \|A^n(T(x))u_n(T(x))\| - \|A^n(T(x))s_n(T(x))\|.$$

In addition, $\|A^n(x)u_n(x)\| = \|A^n(x)\| \geq ce^{\tau n}$ and $\|A^n(x)s_n(x)\| = \|A^n(x)\|^{-1} \leq c^{-1}e^{-\tau n}$ for every $x \in X$, so

$$\begin{aligned} c^{-1}e^{-\tau n} \geq c^{-1}e^{-\tau(n+1)} &\geq \|A^{n+1}(x)s_{n+1}(x)\| \\ &\geq |\sin\beta_n| \cdot \|A^n(T(x))u_n(T(x))\| - \|A^n(T(x))s_n(T(x))\| \\ &\geq |\sin\beta_n| \cdot ce^{\tau n} - c^{-1}e^{-\tau n} \end{aligned}$$

Thus $|\sin\beta_n| \leq 2c^{-2}e^{-2\tau n}$, consequently $\lim_{n\to\infty}\sin\beta_n = 0$ and the vectors $A(x)s(x)$ and $s(T(x))$ are collinear for every $x \in X$. ■

Lemma 3.6. *For any $\tau_0 < \tau$ there exists a positive integer n_0 such that $\|A^n(x)s(x)\| \leq e^{-\tau_0 n}$ for every $x \in M$ and $n \geq n_0$.*

Proof. By the theorem of Furstenberg and Kesten which will be proved in the next subsection, the limit $\lim_{n\to\infty}\dfrac{1}{n}\log\|A^n(x)\|$ exists for almost every point $x \in X$.

Claim 3.7. For μ-almost every $x \in X$ and every T-invariant probability measure μ,

$$\limsup_{n\to\infty}\frac{1}{n}\log\|A^n(x)s(x)\| < -\tau.$$

Proof. Let $\gamma_n := \measuredangle(s(x), s_n(x))$, since $u_n(x)$ and $s_n(x)$ form an orthonormal basis $s(x) = \cos(\gamma_n)s_n(x) + \sin(\gamma_n)u_n(x)$ and

$$\begin{aligned} \|A^n(x)s(x)\| &\leq |\cos\gamma_n| \cdot \|A^n(x)s_n(x)\| + |\sin\gamma_n| \cdot \|A^n(x)u_n(x)\| \\ &\leq \|A^n(x)\|^{-1} + C_1\sum_{m=n}^{\infty} e^{-m\tau}\|A^m(x)\|^{-1}\|A^n(x)\|. \end{aligned}$$

By the theorem of Furstenberg and Kesten (which will be proved in the next section), for every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$-\epsilon \leq \frac{1}{n}\log\|A^n(x)\| - \frac{1}{m}\log\|A^m(x)\| \leq \epsilon, \quad \text{for every } m \geq n \geq n_0.$$

In particular

$$\|A^m(x)\|^{-1}\cdot\|A^n(x)\| \leq \|A^m(x)\|^{-n/m}\cdot\|A^n(x)\| \leq e^{n\epsilon}, \quad \text{for every } m \geq n \geq n_0$$

so by the consequences of Lemma 3.4

$$\begin{aligned}\|A^n(x)s(x)\| &\leq \|A^n(x)\|^{-1} + C_1\sum_{m=n}^{\infty} e^{-m\tau}\|A^m(x)\|^{-1}\|A^n(x)\| \\ &\leq c^{-1}e^{-\tau n} + C_1 e^{n\epsilon}\sum_{m=n}^{\infty} e^{-\tau m} \leq C_1' e^{n\epsilon}e^{-\tau n} \quad \text{for every } n \geq n_0,\end{aligned}$$

where $C_2 = (c^{-1}e^{-n\epsilon} + C_1\sum_{m=0}^{\infty} e^{-\tau m}) < \infty$ depends just on c, C_1 and τ. ■

Let us continue with the proof of Lemma 3.6. By contradiction, we suppose the existence of $\tau_0 < \tau$ such that for every positive integer k there are $n_k \geq k$ and $x_k \in X$ such that

$$\|A^{n_k}(x_k)s(x_k)\| > e^{-\tau_0 n_k}.$$

It follows from the previous inequality that

$$\int_X \phi(x)d\mu_k(x) = \frac{1}{n_k}\sum_{j=0}^{n_k}\log\|A(T^j x_k)s(T^j x_k)\| > -\tau_0,$$

where $\mu_k := n_k^{-1}\sum_{0\leq j\leq n_k-1}\delta_{T^j(x_k)}$ and $\phi(x) := \log\|A(x)s(x)\|$. Since the space of T-invariant probability measures on X is compact with the weak $*$ topology, there is an accumulation point μ of $(\mu_k)_k$ such that $\int_X \phi(x)d\mu(x) \geq -\tau_0$. As $T_*\mu_k = \mu_k + n_k^{-1}(\delta_{T^{n_k}x_k} - \delta_{x_k})$, we get $T_*\mu = \mu$ by taking the limit as k tends to infinite. By Birkhoff's theorem

$$\overline{\phi}(x) := \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\phi(T^j x) = \lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)s(x)\|$$

satisfies $\int_X \overline{\phi}(x)d\mu(x) = \int_X \phi(x)d\mu(x) \geq -\tau_0$. However by Claim 3.7

$$\overline{\phi}(x) \leq \lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)s(x)\| \leq \lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)\| \leq -\tau$$

for almost every point in X. These two inequalities are incompatible since $\tau_0 < \tau$ instead of $\tau \leq \tau_0$. This contradiction proves the lemma. ■

Remark. By Lemma 3.6 there exists $C_3 > 0$ such that $\|A^n(x)s(x)\| \leq C_3 e^{-n\tau}$ for every $x \in X$ and every $n \geq 1$. Analogously, considering backward iterates we construct a unit vector $u(x)$ such that $\|A^{-n}(x)u(x)\| \leq C_3 e^{-n\tau}$ for every $x \in X$ and every $n \geq 1$.

Lemma 3.8. *The vectors $s(x)$ and $u(x)$ are not collinear for every $x \in X$.*

Proof. By Lemma 3.6 there exists $C_3 > 0$ such that $\|A^n(T^{-n}x)s(T^{-n}x)\| \leq C_3 e^{-n\tau}$ for every $x \in X$ and every $n \in \mathbb{N}$. In other words,

$$\|A^{-n}(x)s(x)\| \geq C_3^{-1} e^{n\tau_0},$$

since $A(x)s(x)$ is collinear to $s(T(x))$. By Remark 3.1.1 there exist $u(x)$ and C_3 such that

$$\|A^{-n}(x)u(x)\| \leq C_3 e^{-n\tau_0},$$

for every $x \in X$ and every $n \in \mathbb{N}$. Hence $\|A^{-n}(x)s(x)\| > \|A^{-n}(x)u(x)\|$ for large enough n. In particular $s(x) \neq \pm u(x)$ and the vectors $s(x)$, $u(x)$ cannot be collinear. ■

Finally, we consider E^s_x and E^u_x as the one-dimensional subspaces generated by $s(x)$ and $u(x)$ respectively. The previous lemmas show the $SL(2,\mathbb{R})$-cocycle (T, A) is uniformly hyperbolic. ■

The previous proposition allows us to prove the existence of a large family of uniformly hyperbolic cocycles.

Example. Let $A : X \to SL(2,\mathbb{R})$ be a continuous function on a compact metric space X such that $A(x)$ has positive entries for each $x \in X$. Then any linear cocycle (T, A) defined over a homeomorphism $T : X \to X$ is uniformly hyperbolic.

Let

$$A(x) := \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

where $a(x)d(x) - b(x)c(x) = 1$ for every $x \in X$. Since X is compact and $A(x)$ is continuous, there exists a positive constant δ such that $a(x), b(x), c(x), d(x) \geq \delta$ for every $x \in X$. Let $(v_0, w_0) = v \in \mathbb{R}^2$ an arbitrary unit vector with positive entries and $(v_n, w_n) := A^n(x)v \in \mathbb{R}^2$. As

$$\begin{aligned}
v_1 w_1 &= (a(x)v_0 + b(x)w_0)(c(x)v_0 + d(x)w_0) \\
&= (a(x)c(x)v_0^2 + b(x)d(x)w_0^2 + v_0 w_0(a(x)d(x) + b(x)c(x))) \\
&\geq v_0 w_0(a(x)d(x) + b(x)c(x)) = v_0 w_0(1 + 2b(x)c(x)) \\
&\geq v_0 w_0(1 + 2\delta^2),
\end{aligned}$$

we get by induction that $v_n w_n \geq v_0 w_0 (1+2\delta^2)^n$. Hence

$$\|A^n(x)v\| = \sqrt{v_n^2 + w_n^2} \geq \sqrt{2 v_n w_n} \geq \sqrt{2 v_0 w_0}((1+2\delta^2)^{1/2})^n,$$

which implies

$$\|A^n(x)\| \geq \sup_{\|v\|=1} \sqrt{2 v_0 w_0}((1+2\delta^2)^{1/2})^n \geq ((1+2\delta^2)^{1/2})^n \geq e^{\tau n},$$

where $\tau = \log((1+2\delta^2)^{1/2})$. By Proposition 3.2 we get the cocycle (T, A) is uniformly hyperbolic.

We finish this section by exhibiting a $SL(2,\mathbb{R})$ one-step cocycle which is not uniformly hyperbolic.

Example. Let (T, A) be a $SL(2,\mathbb{R})$ one-step cocycle defined by the left-shift map $T : \{1,2\}^{\mathbb{Z}} \to \{1,2\}^{\mathbb{Z}}$ and the function $A : X \to GL(d,\mathbb{R})$ defined by $A(x) = A_{x_0}$, where

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \text{and} \quad A_2 = R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $q = (\ldots, 1, 1 | 1, 2, 1, 1 \ldots)$ the point in X which has every entry equal to 1 except x_1. Since

$$\lim_{n \to \pm\infty} T^n q = \vec{1} := (\ldots, 1 | 1, 1, \ldots) = T(\vec{1}),$$

by definition q is a homoclinic point for the fixed point $\vec{1}$. In addition $A_2 = R_{\pi/2}$, therefore the cocycle cannot be uniformly hyperbolic. Let us suppose that (T, A) is uniformly hyperbolic, by the invariance of E^s_x,

$$E^s_{T^n q} = A^{2n}(T^{-n} q) E^s_{T^{-n} q} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^n R_{\pi/2} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{n-1} E^s_{T^{-n} q}.$$

Besides, by the continuity of E^s_x we have $E^s_{T^n q} \approx E^s_{\vec{0}} = \{x = 0\}$ for n big enough. Hence, we would have that $E^s_{T^{-n} q} \approx \{y = 0\}$. However, by the continuity of E^s_x, we have $E^s_{T^{-n} q} \approx E^s_{\vec{0}} = \{x = 0\}$, a contradiction.

3.2 Lyapunov exponents

During this section we are going to study the *upper and lower Lyapunov exponents* $\lambda_+(x)$ and $\lambda_-(x)$ of a linear cocycle (T, A).

Definition 3.9. Let (T, A) be a linear cocycle. We define the *upper and lower Lyapunov exponents at a point* $x \in X$ respectively by

$$\lambda_+(x) := \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \quad \text{and} \quad \lambda_-(x) := \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1},$$

whenever the limits exist.

These numbers measure the exponential rates for the norms $\|A^n(x)\|$ and $\|A^n(x)^{-1}\|^{-1}$ of the iterations $A^n(x)$ of a cocycle. We will see in Example 3.2.1 that this limit does not exist for every point $x \in X$. However, the theorem of Furstenberg and Kesten (which is proved during the next subsection) proves that the limits exist in a set of full probability.

During the next subsection we are going to give a more refined interpretation of the Lyapunov exponents of a $SL(2, \mathbb{R})$-cocycle (T, A). We will see that the Lyapunov exponents measure the exponential rates for the norm $\|A^n(x)v\|$ for each vector $v \in \mathbb{R}^2$.

3.2.1 Theorem of Furstenberg and Kesten

During this subsection we are going to work with $GL(d, \mathbb{R})$-linear cocycles. In order to prove the existence of the Lyapunov exponents of a linear cocycle (T, A), we are going to use the following ergodic theorem due to Kingman. We say a sequence of measurable functions $\{\varphi_n\}$ is *subadditive respect to* T if

$$\varphi_{m+n}(x) \leq \varphi_n(T^m x) + \varphi_m(x) \quad \text{for every } m, n \geq 1 \text{ and } x \in X.$$

Theorem 3.10. *(Kingman's Subadditive Ergodic Theorem) Let $\{\varphi_n\}$ be a subadditive sequence of real measurable functions such that $\varphi_1^+ \in L^1(\mu)$. Then φ_n/n converges μ-almost everywhere to some invariant function φ. Moreover, the function φ^+ is integrable and*

$$\int_X \varphi d\mu = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n d\mu = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_X \varphi_n d\mu.$$

Let $\phi : X \to \mathbb{R}$ an arbitrary integrable function. Considering the subadditive sequence $\varphi_n(x) = \sum_{k=0}^{n-1} \phi(T^k x)$ of real measurable functions we get the classic Birkhoff's ergodic theorem. See [AB1] and [V] for a proof of Kingman's subadditive ergodic theorem.

Theorem 3.11. *Let (T, A) be a $GL(d, \mathbb{R})$-linear cocycle and μ be a T-invariant probability measure such that $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$. Then*

$$\lambda_+(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \quad \textit{and} \quad \lambda_-(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1}$$

exist for μ-almost every $x \in X$.

Proof. Let $\varphi_n(x) = \log \|A^n(x)\|$ and $\psi_n(x) = \log \|A^n(x)^{-1}\|$. By hypothesis $\varphi_1^+ = \log^+ \|A(x)\| \in L^1(\mu)$ and $\psi_1^+ = \log^+ \|A(x)^{-1}\| \in L^1(\mu)$. Moreover,

$$\begin{aligned}\varphi_{n+m}(x) = \log \|A^{n+m}(x)\| = \log \|A^n(T^m x) \cdot A^m(x)\| \leq \log \|A^n(T^m x)\| \cdot \|A^m(x)\| \\ \leq \log \|A^n(T^m x)\| + \log \|A^m(x)\| = \varphi_n(T^m x) + \varphi_m(x)\end{aligned}$$

proves that $\{\varphi_n\}_{n\in\mathbb{N}}$ is a subadditive sequence of functions. Analogously $\{\psi_n\}_{n\in\mathbb{N}}$ is another subadditive sequence of functions. Hence, by Kingman's subadditive ergodic theorem we obtain the conclusion. ∎

Remark. Note that this theorem generalizes the ergodic Birkhoff's theorem which says that for n large enough

$$\varphi(T^{n-1}(x)) + \varphi(T^{n-1}(x)) + \cdots + \varphi(x) \simeq cn,$$

where $c = \int_X T(x) d\mu(x)$. In fact, the Theorem 3.11 says that

$$\|A(T^{n-1}x)A(T^{n-2}x) \cdots A(x)\| \simeq e^{\lambda_+(x)n},$$

for n large enough. It shows that theorem of Furstenberg and Kesten generalizes the ergodic Birkhoff's theorem for matrix products.

We proceed to show some basic examples of linear cocycles and their Lyapunov exponents. The following examples come from the notes [AB2], and the books [V] and [P].

Example. Let us suppose that $A : X \to GL(d, \mathbb{R})$ is a constant function. Then $\lambda_+(x)$ is equal to the spectral radius $\rho(A)$ of the matrix A for every $x \in X$. In fact, the Gelfand's formula says that $\lim_{n\to\infty} \|A^n\|^{1/n} = \rho(A)$, so

$$\lambda_+(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n\| = \log \rho(A).$$

In the following two examples and the rest of the thesis we are going to deal with $SL(2, \mathbb{R})$ cocycles. In particular, the next remark will be used many times.

Remark. Since $\|M\| = \|M^{-1}\|$ for every M in $SL(2, \mathbb{R})$, we have

$$\lambda_-(x) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)^{-1}\|^{-1} = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|^{-1} = -\lambda_+(x),$$

for all the points $x \in X$ where the limit $\lambda_+(x)$ exists.

Now, we exhibit two derivative cocycles which have constants Lyapunov exponents.

Example. Let $T : \mathbb{T}^2 \to \mathbb{T}^2$ be a diffeomorphism induced by the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x+\alpha, y+\beta)$. Since $DT_x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for every $x \in \mathbb{T}^2$,

$$\lambda_+(x) = \lim_{n\to\infty} \frac{\log \|D_x(T^n)\|}{n} = \log \rho(I) = \log 1 = 0$$

for every $x \in \mathbb{T}^2$. Hence $\lambda_+(x) = 0 = \lambda_-(x)$ for every $x \in \mathbb{T}^2$.

Now, we are going to prove that the Anosov diffeomorphism presented on Example 2.2 have non-zero constants Lyapunov exponents.

Example. Let us consider the Anosov diffeomorphism presented on Example 2.2. By Example 3.2.1 this derivative cocycle has Lyapunov exponent

$$\lambda_+(x) = \lim_{n\to\infty} \frac{\log \|D_x(T^n)\|}{n} = \log \rho(A) = \log \left(\frac{3+\sqrt{5}}{2}\right),$$

for every $x \in \mathbb{T}^2$. Since $A \in SL(2,\mathbb{R})$ we get

$$\lambda_-(x) = -\lambda_+(x) = -\log \left(\frac{3+\sqrt{5}}{2}\right),$$

for every $x \in \mathbb{T}^2$.

The next corollary gives a more detailed description about the set where the limits $\lambda_+(x)$ and $\lambda_-(x)$ exist.

Corollary 3.12. *For every linear cocycle (T, A) there is a Borel set of full probability $\mathcal{R}$ such that $\lambda_+(x)$ and $\lambda_-(x)$ exist for every $x \in \mathcal{R}$.*

Proof. By hypothesis X is compact and A is a continuous function, hence $\log^+ \|A^{\pm 1}\| \in L^1(\mu)$ for every T-invariant probability measure μ. So the limits $\lambda_+(x)$ and $\lambda_-(x)$ exist for every x in a set of full probability. Moreover, the set where the limit $\lambda_+(x) := \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\|$ exists is Borel since it is the limit of a sequence of Borel measurable functions $\frac{1}{n} \log \|A^n(x)\|$. Analogously, the set where $\lambda_-(x)$ exists is also Borel, so we get the result. ■

We call *regular point* to every element $x \in \mathcal{R}$. In the following, we prove the existence of a large class of regular points for cocycles where the function T has many periodic points.

Proposition 3.13. *Let (T, A) be a $GL(d, \mathbb{R})$-linear cocycle, then every periodic point $p = T^n p$ is a regular point.*

Proof. On the one hand

$$\begin{aligned}
\frac{1}{nk+r} \log \|A^{nk+r}(p)\| &= \frac{1}{nk+r} \log \|A^r(T^{nk}p)A^{nk}(p)\| \\
&\leq \frac{1}{nk+r} \log \|A^r(p)\| + \frac{1}{nk+r} \log \|A^{nk}(p)\| \\
&\leq \frac{C(p)}{nk+r} + \frac{1}{nk+r} \log \|A^n(p)^k\| \\
&= \frac{C(p)}{nk+r} + \frac{1}{n} \log \|A^n(p)^k\|^{1/k} \to \frac{1}{n} \log \rho(A^n(p))
\end{aligned}$$

for every $r \in \{0, 1, \ldots, n-1\}$ where $C(p) = \max_{x \in X}\{\|A(x)\|, \|A^2(x)\|, \ldots \|A^{n-1}(x)\|\}$ is a positive constant depending on p. On the other hand

$$\begin{aligned}
\frac{1}{nk+r} \log \|A^{nk+r}(p)\| &= \frac{1}{nk+r} \log \|A^{n-r}(T^{nk+r}p)^{-1}A^{n(k+1)}(p)\| \\
&\geq \frac{1}{nk+r} \log \|A^{n(k+1)}(p)\| - \frac{1}{nk+r} \log \|A^{n-r}(T^r p)\| \\
&\geq \frac{nk+n}{nk+r} \cdot \frac{1}{n(k+1)} \log \|A^n(p)^{k+1}\| - \frac{C(p)}{nk+r} \\
&= \frac{1}{n} \log \|A^n(p)^{k+1}\|^{1/(k+1)} - \frac{C(p)}{nk+r} \to \frac{1}{n} \log \rho(A^n(p))
\end{aligned}$$

So,

$$\lambda_+(p) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(p)\| = \frac{1}{n} \log \rho(A^n(p)).$$

A similar calculation shows that

$$\lambda_-(p) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(p)^{-1}\|^{-1} = \frac{1}{n} \log \rho(A^n(p)^{-1})^{-1}.$$

■

Although the previous example exhibit a large set of points which are regular, it is normal to find points $x \in X$ for which the Lyapunov exponents $\lambda_+(x)$ and $\lambda_-(x)$ do not exist.

Example. Let (T, A) be the $SL(2, \mathbb{R})$ one-step cocycle of Example 3.1.1. It is defined by the left-shift map $T : \{1, 2\}^{\mathbb{Z}} \to \{1, 2\}^{\mathbb{Z}}$ and the function $A : X \to GL(d, \mathbb{R})$ defined by $A(x) = A_{x_0}$, where

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \text{and} \quad A_2 = R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can construct a point x which is composed by series of sequences with the same entries. Each sequence which has only the entry 2 will have a long multiple of four because $A_2^4 = R_{\pi/2}^4 = I$, so the product will not change the value of $A^n(x)$ when n increases. By considering a point x such that the large of the sequences with the same entries increase quickly, we are going to get a point x such that 0 and $\log 2$ are two accumulation points of the sequence $\left(\frac{1}{n}\log\|A^n(x)\|\right)_{n\in\mathbb{N}}$ so the limit $\lambda_+(x)$ does not exist for such point. Moreover, by choosing properly the entries $x_1, x_2, \ldots, x_n \ldots$ we can construct several points for which $\lambda_+(x)$ does not exist. Similarly, by choosing properly the entries $x_{-1}, x_{-2}, \ldots, x_{-n} \ldots$ we can construct several points for which $\lambda_-(x)$ does not exist. Hence, by choosing properly the entries of $x = (x_n)_{n\in\mathbb{Z}}$ we can construct several points for which $\lambda_+(x)$ and $\lambda_-(x)$ do not exist.

In the following, we show a cocycle which is not uniformly hyperbolic, nevertheless its unique Lyapunov exponent λ_+ is positive. This example was exhibited by Herman in [He].

Example. (Herman's formula) Let $T : S^1 \to S^1$ be an irrational rotation. Let $A : S^1 \to SL(2,\mathbb{R})$ be a function defined by $A(x) = A_0 \cdot R_{2\pi x}$ where

$$A_0 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \quad \text{and} \quad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Then

$$\lambda_+ \geq \log\left(\frac{\sigma + \sigma^{-1}}{2}\right),$$

where λ_+ is the upper Lyapunov exponent respect to the unique T-invariant probability measure for this cocycle.

Let $\omega \in 2\pi(\mathbb{R} \setminus \mathcal{Q})$ the angle of rotation of $T : S^1 \to S^1$. By the theorem of Furstenberg and Kesten,

$$\begin{aligned}\lambda_+ &= \lim_{n\to\infty} \frac{1}{n}\int_{S^1} \log\|A^n(y)\|dm \\ &= \lim_{n\to\infty} \frac{1}{2\pi n}\int_0^{2\pi} \log\|A_0 R_{x+(n-1)\omega}\cdots A_0 R_{x+\omega} A_0 R_x\|dx.\end{aligned}$$

Let us consider the function $R : \mathbb{C} \to M_2(\mathbb{C})$ defined by

$$R(z) = \begin{pmatrix} (z^2+1)/2 & -(z^2-1)/2i \\ (z^2-1)/2i & (z^2+1)/2 \end{pmatrix}$$

which takes values in the set of matrices with complex coefficients $M_2(\mathbb{C})$. Note that $R(e^{i\theta}) = e^{i\theta}R_\theta$ for every $\theta \in \mathbb{R}$. Let $C_n : \mathbb{C} \to M_2(\mathbb{C})$, defined by

$$C_n(z) = A_0R(e^{(n-1)wi}z)\cdots A_0R(e^{wi}z)A_0R(z),$$

where $\|\cdot\|$ is the norm of the maximum absolute value of the entries. Since $C_n(e^{ix}) = e^{i\tau}A^n(x)$ with $\tau = nx + n(n-1)\omega/2$,

$$\lambda_+ = \lim_{n\to\infty} \frac{1}{2\pi n}\int_0^{2\pi} \log\|C_n(e^{ix})\|dx.$$

Moreover, $\log\|C_n(z)\|$ is subharmonic since the $\log$ of the absolute value of a holomorphic function is subharmonic and the maximum of subharmonic functions is subharmonic. Hence,

$$\begin{aligned}\lambda_+ &\geq \lim_{n\to\infty}\frac{1}{n}\log\|C_n(0)\| = \lim_{n\to\infty}\frac{1}{n}\log\|(A_0R(0))^n\| \\ &= \rho(A_0R(0)) = \log\left(\frac{\sigma+\sigma^{-1}}{2}\right).\end{aligned}$$

Remark. Actually, the formula proved by A.Avila and J.Bochi in [AB3] shows that

$$\lambda_+ = \log\left(\frac{\sigma+\sigma^{-1}}{2}\right).$$

Remark. This example exhibit a cocycle for which its unique Lyapunov exponent is not zero, nevertheless it is not uniformly hyperbolic. Assume (T, A) be a uniformly hyperbolic cocycle with stable direction E^s. It can be proved that there exist integers t, a and e such that $T : S^1 \to S^1$, $A : S^1 \to SL(2,\mathbb{R})$ and $E^s : S^1 \to \mathbb{P}^1$ are homotopic to the mappings $x \mapsto tx$, $x \mapsto R_{2\pi ax}$ and $x \mapsto \mathbb{R}(\cos(\pi ex), \sin(\pi ex))$ respectively. By the invariance of E^s we get $A(x)\cdot E^s_x = E^s_{T(x)}$, in particular $2a + e = te$. Since T is an irrational rotation $t = e = 1$, so there is no solution to the equation $2a + e = te$. Hence the cocycle (T, A) of Example 3.2.1 is not uniformly hyperbolic.

3.2.2 Theorem of Oseledets

During this subsection we are going to prove the Theorem of Oseledets for $SL(2,\mathbb{R})$-cocycles. We follow the proofs given by A. Avila and J. Bochi in [AB2] and the proof in [V] page 30.

Theorem 3.14. *(Theorem of Oseledets) Let $T : X \to X$ be a mapping which preserves a probability measure μ. Let $A : X \to SL(2,\mathbb{R})$ be a function such that $\log\|A(x)\| \in L^1(\mu)$.*

For almost every $x \in X$ either $\lambda_+(x) = \lambda_-(x)$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_-(x) = \lambda_+(x), \quad \textit{for every } v \in \mathbb{R}^2,$$

or $\lambda_+(x) > \lambda_-(x)$ and there exists a one dimensional vector space E_x^- such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| = \begin{cases} \lambda_+(x) & \textit{if } v \in \mathbb{R}^2 \setminus E_x^-, \\ \lambda_-(x) & \textit{if } v \in E_x^- \setminus \{0\}. \end{cases}$$

Moreover, the spaces E_x^- are invariant and depend measurably on the point x.

Proof. Firstly, suppose that $\lambda_+(x) = -\lambda_-(x) = 0$ for some $x \in X$. By the submultiplicativity of the norm

$$\|A^n(x)\|^{-1} \cdot \|v\| = \|A^n(x)^{-1}\|^{-1} \cdot \|v\| \leq \|A^n(x)v\| \leq \|A^n(x)\| \cdot \|v\|,$$

for any vector $v \in \mathbb{R}^2$. As a result,

$$\frac{1}{n} \log \|A^n(x)\|^{-1} \cdot \|v\| \leq \frac{1}{n} \log \|A^n(x)v\| \leq \frac{1}{n} \log \|A^n(x)\| \cdot \|v\|,$$

which implies

$$0 = \lambda_-(x) \cdot \|v\| \leq \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \leq \lambda_+(x) \cdot \|v\| = 0.$$

Secondly, suppose that $\lambda_+(x) > 0$ for some $x \in X$. We are going to prove this case by dividing it in five lemmas.

By the theorem of Furstenberg and Kesten $\|A^n(x)\| > 1$ for n large enough. By Lemma 3.3 the vectors $u_n(x)$ and $s_n(x)$ form an orthonormal basis such that

$$\|A^n(x)u_n(x)\| = \|A^n(x)\| \quad \text{and} \quad \|A^n(x)s_n(x)\| = \|A^n(x)\|^{-1}.$$

Lemma 3.15.

$$\lim_{n\to\infty} \frac{1}{n} \log |\measuredangle(s_n(x), s_{n+1}(x))| \leq -2\lambda(x) \quad \textit{for every } x \in X.$$

Proof. By definition of $s_n(x)$

$$\|A^{n+1}(x)s_n(x)\| \leq \|A(T^n x)\| \cdot \|A^n(x)s_n(x)\| = \|A(T^n x)\| \cdot \|A^n(x)\|^{-1}.$$

Let $\alpha_n := \measuredangle(s_n(x), s_{n+1}(x))$, since $u_{n+1}(x)$ and $s_{n+1}(x)$ form an orthonormal basis we get $s_n(x) = \sin(\alpha_n)u_{n+1}(x) + \cos(\alpha_n)s_{n+1}(x)$. Thus

$$\|A^{n+1}(x)s_n(x)\| \geq \|\sin(\alpha_n)A^{n+1}(x)u_{n+1}(x)\| = |\sin(\alpha_n)| \cdot \|A^{n+1}(x)\|.$$

By the previous two inequalities,

$$|\sin(\alpha_n)| \leq \frac{\|A(T^n x)\|}{\|A^n(x)\| \cdot \|A^{n+1}(x)\|}.$$

By the hypothesis and Lemma 3.20 the limit $\lim_{n\to\infty} \log \|A(T^n x)\|/n$ is equal to 0. Hence

$$\limsup_{n\to\infty} \frac{1}{n}\log(\alpha_n) \leq \lim_{n\to\infty} \frac{1}{n}\log\|A(T^n x)\| - \lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)\| - \lim_{n\to\infty}\frac{1}{n}\log\|A^{n+1}(x)\| = -2\lambda_+(x).$$

■

Lemma 3.16. *The sequence $(s_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence.*

Proof. From Lemma 3.15 we get that $|\sin(\alpha_n)| \leq e^{n(-2\lambda(x)+\epsilon)}$ for every n large enough. Up to replacing s_n by $-s_n$ we can assume that $\sin(\alpha_n) \geq 0$ for every n large enough. So

$$\begin{aligned}\|s_{n+1}(x) - s_n(x)\|^2 &\leq (s_{n+1}(x) - s_n(x), s_{n+1}(x) - s_n(x)) \\ &= 2(1-\cos(\alpha_n)) = 2(1-\sqrt{1-e^{2n(-2\lambda(x)+\epsilon)}}) \\ &\leq 2e^{2n(-2\lambda(x)+\epsilon)}.\end{aligned}$$

More generally, we get

$$\begin{aligned}\|s_{n+k}(x) - s_n(x)\| &\leq \|s_{n+k}(x) - s_{n+k-1}(x)\| + \cdots + \|s_{n+1}(x) - s_n(x)\| \\ &\leq \sqrt{2}e^{n(-2\lambda(x)+\epsilon)(n+k)} + \cdots + \sqrt{2}e^{n(-2\lambda(x)+\epsilon)} \\ &\leq \sqrt{2}e^{n(-2\lambda(x)+\epsilon)}(e^{(-2\lambda(x)+\epsilon)k} + \cdots + 1) \\ &\leq C_2 e^{n(-2\lambda(x)+\epsilon)},\end{aligned}$$

where $C_2 = \sqrt{2}\sum_{k=0}^{\infty} e^{(-2\lambda(x)+\epsilon)k}$ for small enough ϵ. Hence, $\{s_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of unit vectors and limit $\lim_{n\to\infty} s_n(x)$ is well defined. In the following we denote $s(x) := \lim_{n\to\infty} s_n(x)$.

By Lemma 3.15

$$|\sin\measuredangle(s_{n+k}(x), s_n(x))| \leq \sum_{m=n}^{n+k-1} |\sin\measuredangle(s_{m+1}(x), s_m(x))| \leq C_1 \sum_{m=n}^{n+k-1} e^{-m\lambda_+(x)}\|A^m(x)\|^{-1}.$$

Consequently

$$|\sin\measuredangle(s(x),s_n(x))|\le C_1\sum_{m=n}^{\infty}e^{-m\tau}\|A^m(x)\|^{-1}\le C_2\sum_{m=n}^{\infty}e^{-2m\lambda_+(x)}.$$

In particular

$$\limsup_{n\to\infty}\frac{1}{n}\log|\sin(\gamma_n)|\le -2\lambda_+(x),$$

where $\gamma_n := \measuredangle(s(x),s_n(x))$. ■

Lemma 3.17. *The vector $s(x)$ satisfies*

$$\lim_{n\to\infty}\frac{1}{n}\log\|A^n(x)s(x)\| = -\lambda_+(x).$$

Proof. On the one hand $\log\|A^n(x)\|^{-1}\le\log\|A^n(x)s(x)\|$ implies

$$-\lambda_+(x)=\liminf_{n\to\infty}\frac{1}{n}\log\|A^n(x)\|^{-1}\le\liminf_{n\to\infty}\frac{1}{n}\log\|A^n(x)s(x)\|.$$

On the other hand we are going to prove that $\limsup_{n\to\infty}\dfrac{1}{n}\log\|A^n(x)s(x)\|\le-\lambda_+(x)$. This will finish the proof of the lemma.

Since $\gamma_n := \measuredangle(s(x),s_n(x))$ and the pair $u_n(x),s_n(x)$ form an orthonormal basis $s(x)=\cos(\gamma_n)s_n(x)+\sin(\gamma_n)u_n(x)$ and

$$\|A^n(x)s(x)\|\le|\cos(\gamma_n)|\cdot\|A^n(x)s_n(x)\|+|\sin(\gamma_n)|\cdot\|A^n(x)u_n(x)\|.$$

Since

$$\limsup_{n\to\infty}\frac{1}{n}\log(a_n+b_n)=\max\Big\{\limsup_{n\to\infty}\frac{1}{n}\log(a_n),\limsup_{n\to\infty}\frac{1}{n}\log(b_n)\Big\},$$

for any pair of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ we get

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\log\|A^n(x)s(x)\|\\
&=\max\Big\{\limsup_{n\to\infty}\frac{1}{n}\log|\cos(\gamma_n)|\cdot\|A^n(x)s_n(x)\|,\limsup_{n\to\infty}\frac{1}{n}\log|\sin(\gamma_n)|\cdot\|A^n(x)u_n(x)\|\Big\}\\
&=\max\Big\{\limsup_{n\to\infty}\frac{1}{n}\log\|A^n(x)\|^{-1},\limsup_{n\to\infty}\frac{1}{n}\log|\sin(\gamma_n)|+\frac{1}{n}\log\|A^n(x)\|\Big\}\\
&\le\max\{-\lambda_+(x),-2\lambda_+(x)+\lambda_+(x)\}=-\lambda_+(x)
\end{aligned}$$

by the comment previous to this lemma. ■

Lemma 3.18. *Let $v \in \mathbb{R}^2$ be a vector which does not belong to the one-dimensional subspace generated by $s(x)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_+(x).$$

Proof. On the one hand

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \leq \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)\| \cdot \|v\| = \lambda_+(x)$$

On the other hand, we are going to prove that $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \geq \lambda_+(x)$ for almost every $x \in X$. Let $\delta_n := \measuredangle(v, s_n(x))$ so $v = \cos(\delta_n)s_n(x) + \sin(\delta_n)u_n(x)$ and

$$\begin{aligned}
\|A^n(x)v\| &\geq |\sin\delta_n| \cdot \|A^n(x)u_n(x)\| - |\cos\delta_n| \cdot \|A^n(x)s_n(x)\| \\
&= |\sin\delta_n| \cdot \|A^n(x)\| - |\cos\delta_n| \cdot \|A^n(x)\|^{-1}.
\end{aligned}$$

By the Theorem of Furstenberg and Kesten for every $c < 1$ we have $\|A^n(x)\| \geq e^{cn\lambda_+(x)}$ for n large enough. In particular,

$$\begin{aligned}
\frac{1}{n} \log \left| |\sin\delta_n| \cdot \|A^n(x)\| - |\cos\delta_n| \cdot \|A^n(x)\|^{-1} \right| &\geq \frac{1}{n} \log \left| |\sin\delta_n| e^{cn\lambda_+(x)} - |\cos\delta_n| \cdot e^{-cn\lambda_+(x)} \right| \\
&= \frac{1}{n} \log \left| e^{cn\lambda_+(x)} \cdot (|\sin\delta_n| - |\cos\delta_n| \cdot e^{-2cn\lambda_+(x)}) \right| \\
&= c\lambda_+(x) + \frac{1}{n} \log \left| |\sin\delta_n| - |\cos\delta_n| \cdot e^{-2cn\lambda_+(x)} \right|
\end{aligned}$$

By hypothesis v is not collinear to $s(x)$ and since $\lambda_+(x) > 0$ the second term tends to zero in the limit, so

$$\liminf_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \geq c\lambda_+(x)$$

for every positive constant $c < 1$. In particular,

$$\liminf_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| \geq \lambda_+(x).$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x)v\| = \lambda_+(x)$$

for every vector $v \in \mathbb{R}^2$ which does not belong to the one-dimensional subspace generated by $s(x)$. ∎

Lemma 3.19. *The vectors $A(x)s(x)$ and $s(T(x))$ are collinear.*

Proof. On the one hand, by Lemma 3.17

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(T(x))A(x)s(x)\| = \lim_{n\to\infty} \frac{1}{n+1} \log \|A^{n+1}(x)s(x)\| = -\lambda_+(x).$$

On the other hand

$$\lim_{n\to\infty}\frac{1}{n}\log\|A^n(T(x))v\| = \lambda_+(x),$$

for every $v \in \mathbb{R}^2$ which is not collinear to $s(T(x))$. Hence, the vectors $A(x)s(x)$ and $s(T(x))$ are collinear. ■

Let E_x^- be the one-dimensional subspace generated by $s(x)$. The previous lemmas contain all the claims in Oseledets Theorem. ■

Remark. Note that when the cocycle (T, A) is uniformly hyperbolic cocycle we have $E_x^- = E_x^s$ for each $x \in X$. This follows from the uniqueness of E_x^s.

Remark. The previous theorem can be extended to $GL(d, \mathbb{R})$-linear cocycles. See [V] page 40 for the general version of the Theorem of Oseledets. See [F] for a discussion about the Theorem of Oseledets and its subsequent generalizations.

We conclude this subsection with the proof of the following auxiliar lemma which was used on the proof of the Theorem of Oseledets.

Lemma 3.20. *Let $\phi : X \to \mathbb{R}$ be a measurable function such that ϕ is integrable respect to a T-invariant probability measure μ. Then*

$$\lim_{n\to\infty}\frac{1}{n}\phi(T^n x) = 0 \quad \textit{for almost every } x \in X.$$

Proof. Let $\epsilon > 0$. Since μ is a T-invariant measure

$$\begin{aligned}\mu(\{x \in X : |\phi(T^n(x))| \geq n\epsilon\}) &= \mu(\{x \in X : |\phi(x)| \geq n\epsilon\}) \\ &= \sum_{k=n}^{\infty}\mu(\{x \in X : k \leq \frac{|\phi(x)|}{\epsilon} \leq k+1\}).\end{aligned}$$

So we get

$$\begin{aligned}\sum_{n=1}^{\infty}\mu(\{x \in X : |\phi(T^n(x))| \geq n\epsilon\}) &= \sum_{k=1}^{\infty}k\mu(\{x \in X : k \leq \frac{|\phi(x)|}{\epsilon} \leq k+1\}) \\ &\leq \int_X |\phi(x)|du(x).\end{aligned}$$

As ϕ is integrable all the previous sums are finite. In particular, the set $B(\epsilon)$ of points x such that $|\phi(T^n(x))| \geq n\epsilon$ for infinite positive integers n has zero measure. Note that for each $x \in X \setminus B(\epsilon)$ we get $|\phi(T^n(x))| < n\epsilon$ for every n large enough. Let us consider the set $B = \bigcup_{i=1}^{\infty} B(1/i)$ of zero measure. For each $x \in X \setminus B$ and $\epsilon > 0$ there exists a positive integer $i \geq 1$ such that $1/i < \epsilon$ and consequently $|\phi(T^n(x))|/n < 1/i < \epsilon$ for every n large enough. So $\lim_{n\to\infty}\dfrac{1}{n}\phi(T^n x) = 0$ for almost every $x \in X$. ■

3.3 Fiber-bunching condition

Now, we are going to define the fiber-bunching condition, the main assumption in Theorem 1.5.

Definition 3.21. Let $T : X \to X$ be either a subshift of finite type or an Anosov diffeomorphism. A linear cocycle (T, A) is called *fiber-bunched* if there exists $\alpha > 0$ such that the function $A : X \to GL(d, \mathbb{R})$ is α-Hölder and for every $x \in X$

$$\|A(x)\| \cdot \|A(x)^{-1}\| \cdot 2^{-\alpha} < 1$$

in the case where T is a subshift of finite type and

$$\|A(x)\| \cdot \|A(x)^{-1}\| \cdot \lambda^{\alpha} < 1$$

in the case where T is an Anosov diffeomorphism. We also say that the linear cocycle (T, A) satisfies the *fiber-bunching condition*.

Remark. In our context, A takes values in $SL(2, \mathbb{R})$. Since $\|M\| = \|M^{-1}\|$ for every M in $SL(2, \mathbb{R})$, we can write the fiber bunching condition as

$$\|A(x)\|^2 \cdot 2^{-\alpha} < 1 \quad \text{or} \quad \|A(x)\|^2 \cdot \lambda^{\alpha} < 1,$$

if the cocycle is considered over a subshift of finite type or an Anosov diffeomorphism respectively.

The family of fiber bunched cocycles have shown to be very interesting. I highlight the theorem proved in [BBB] in dimension 2. It says the Lyapunov exponents vary continuously when restricted to the subset of fiber-bunched $GL(2, \mathbb{R})$-cocycles over a subshift of finite type.

Example. Let $T : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ be the left shift map. Let $A_\sigma : \{0, 1\}^{\mathbb{Z}} \to SL(2, \mathbb{R})$ defined by $A(x) = A_{x_0}$ where

$$A_0 := \begin{pmatrix} \sigma^{-1} & 0 \\ 0 & \sigma \end{pmatrix} \quad and \quad A_1 := \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix},$$

and σ is a positive constant greater than 1. By definition the $SL(2, \mathbb{R})$-cocycle (T, A_σ) is fiber-bunched if and only if $\sigma^2 < 2^{\alpha}$.

The most useful property of fiber-bunched cocycles is the existence of holonomies. The following theorem proved by C. Bonatti, X. Gómez-Mont and M. Viana in [BGMV] (see also [KS]) gives the existence of these maps and describes their main properties.

Theorem 3.22. *Let (T, A) be a fiber-bunched linear cocycle. For every $y \in W^s(x)$, the limit*

$$H^s_{x\leftarrow y} := \lim_{n\to\infty} A^n(x)^{-1}A^n(y)$$

exists and defines a linear isomorphism $H^s_{x\leftarrow y} : \mathbb{R}^d \to \mathbb{R}^d$. We say that the family of linear automorphisms $\{H^s_{x\leftarrow y} : y \in W^s(x)\}$ is the stable holonomy *for the cocycle (T, A). Besides, for every $y, z \in W^s(x)$*

$$H^s_{x\leftarrow x} = I, \quad H^s_{x\leftarrow y} = H^s_{x\leftarrow z} \cdot H^s_{z\leftarrow y},$$

$$A(x) \cdot H^s_{x\leftarrow y} = H^s_{Tx\leftarrow Ty} \cdot A(y).$$

Also, for every $y \in W^s_{loc}(x)$, there is a positive constant C_0 such that $\|H^s_{x\leftarrow y} - I\| \le C_0 d(x, y)^\alpha$. Finally, if $y \in W^u(x)$ there are analogous properties for

$$H^u_{x\leftarrow y} := \lim_{n\to\infty} A^{-n}(x)^{-1}A^{-n}(y).$$

Proof. Let us consider that (T, A) is a linear cocycle over a subshift of finite type. For every $y \in W^s_{loc}(x)$ we have $d(T^n x, T^n y) \le 2^{-n}d(x, y)$ for every positive integer n . Hence,

$$\begin{aligned}
\|A^{n+1}(x)^{-1}A^{n+1}(y) - A^n(x)^{-1}A^n(y)\| &\le \|A^n(x)^{-1}\| \cdot \|A(T^n x)^{-1}A(T^n y) - I\| \cdot \|A^n(y)\| \\
&\le 2^{n\alpha} \cdot \|A(T^n x)^{-1}A(T^n y) - I\| \\
&\le 2^{n\alpha} \cdot \|A(T^n x)^{-1}\| \cdot \|A(T^n y) - A(T^n x)\| \\
&\le C2^{n\alpha} d(T^n x, T^n y)^\alpha = C2^{n(\alpha-1)}d(x, y)^\alpha.
\end{aligned}$$

By the triangular inequality

$$\begin{aligned}
\|A^{n+k}(x)^{-1}A^{n+k}(y) - A^n(x)^{-1}A^n(y)\| &\le \sum_{j=n}^{n+k-1} \|A^{j+1}(x)^{-1}A^{j+1}(y) - A^j(x)^{-1}A^j(y)\| \\
&\le C \sum_{j=n}^{n+k-1} 2^{j(\alpha-1)}d(x, y)^\alpha \le C \sum_{j=n}^{\infty} 2^{j(\alpha-1)}d(x, y)^\alpha,
\end{aligned}$$

for every positive integer n. As a result, $\{A^n(x)^{-1}A^n(y)\}$ is a Cauchy sequence and the limit $H^s_{x\leftarrow y} := \lim_{n\to\infty} A^n(x)^{-1}A^n(y)$ exists. Moreover,

$$\begin{aligned}
\|A^n(x)^{-1}A^n(y) - I\| &\le \sum_{m=0}^{n-1} \|A^{m+1}(x)^{-1}A^{m+1}(y) - A^m(x)^{-1}A^m(y)\| \\
&\le C \sum_{m=0}^{n-1} 2^{m(\alpha-1)}d(x, y)^\alpha \le C \sum_{m=0}^{\infty} 2^{m(\alpha-1)}d(x, y)^\alpha,
\end{aligned}$$

for every positive integer n. In particular, there is a positive constant $C_0 := C\sum_{m=0}^{\infty} 2^{m(\alpha-1)}$ such that $\|H^s_{x\leftarrow y} - I\| \leq C_0 d(x,y)^\alpha$ for every $y \in W^s_{loc}(x)$. In addition,

$$H^s_{x\leftarrow x} = \lim_{n\to\infty} A^n(x)^{-1}A^n(x) = I \quad \text{for every } x \in X\text{, and}$$

$$\begin{aligned} H^s_{x\leftarrow y} &= \lim_{n\to\infty} A^n(x^{-1}A^n(y) = \lim_{n\to\infty} A^n(x)^{-1}A^n(z)\cdot A^n(z)^{-1}A^n(y) \\ &= \lim_{n\to\infty} A^n(x)^{-1}A^n(z)\cdot \lim_{n\to\infty} A^n(z)^{-1}A^n(y) = H^s_{x\leftarrow z}\cdot H^s_{z\leftarrow y} \end{aligned}$$

for every $z, y \in W^s_{loc}(x)$. By the previous two properties the limit $\lim_{n\to\infty} A^n(x)^{-1}A^n(y)$ is well-defined for every $y \in W^s(x)$. Finally,

$$\begin{aligned} A(z)\cdot H^s_{z\leftarrow y} = A(z)\cdot \lim_{n\to\infty} A^n(z)^{-1}A^n(y) &= \lim_{n\to\infty} A^{n-1}(Tz)^{-1}A^n(y) \\ &= \lim_{n\to\infty} A^{n-1}(Tz)^{-1}A^{n-1}(Ty)A(y) = H^s_{Tz\leftarrow Ty}\cdot A(y). \end{aligned}$$

Now, let us consider that (T, A) is a linear cocycle over an Anosov diffeomorphism. For every $y \in W^s_{loc}(x)$ we have $d(T^n x, T^n y) \leq \lambda^n d(x,y)$ for every positive integer n . Hence,

$$\begin{aligned} \|A^{n+1}(x)^{-1}A^{n+1}(y) - A^n(x)^{-1}A^n(y)\| &\leq \|A^n(x)^{-1}\|\cdot\|A(T^n x)^{-1}A(T^n y) - I\|\cdot\|A^n(y)\| \\ &\leq \lambda^{-n\alpha}\cdot\|A(T^n x)^{-1}A(T^n y) - I\| \\ &\leq \lambda^{-n\alpha}\cdot\|A(T^n x)^{-1}\|\cdot\|A(T^n y) - A(T^n x)\| \\ &\leq C\lambda^{-n\alpha}d(T^n x, T^n y)^\alpha = C\lambda^{n(1-\alpha)}d(x,y)^\alpha. \end{aligned}$$

By the triangular inequality

$$\begin{aligned} \|A^{n+k}(x)^{-1}A^{n+k}(y) - A^n(x)^{-1}A^n(y)\| &\leq \sum_{j=n}^{n+k-1} \|A^{j+1}(x)^{-1}A^{j+1}(y) - A^j(x)^{-1}A^j(y)\| \\ &\leq C\sum_{j=n}^{n+k-1} \lambda^{j(1-\alpha)}d(x,y)^\alpha \leq C\sum_{j=n}^{\infty} \lambda^{j(1-\alpha)}d(x,y)^\alpha, \end{aligned}$$

for every positive integer n. As a result, $\{A^n(x)^{-1}A^n(y)\}$ is a Cauchy sequence and the limit $H^s_{x\leftarrow y} := \lim_{n\to\infty} A^n(x)^{-1}A^n(y)$ exists. The other properties follow directly as we proved for a subshift of finite type. Moreover, the existence and the properties of $H^u_{z\leftarrow y} := \lim_{n\to\infty} A^{-n}(z)^{-1}A^{-n}(y)$ follow analogously. ■

Example. Let us consider the one-step cocycle defined on Example 3.1. In this case the holonomies always exist. For every $y \in W^s(x)$ there exists n_0 such that $x_n = y_n$ for all

$n \geq n_0$. As a result

$$\begin{aligned} H^s_{x \leftarrow y} = \lim_{n \to \infty} A^n(x)^{-1} A^n(y) = \lim_{n \to \infty} (A_{x_{n-1}} \dots A_{x_0})^{-1} (A_{y_{n-1}} \dots A_{y_0}) \\ = A^{-1}_{x_0} \dots A^{-1}_{x_{n_0-1}} A_{y_{n_0-1}} \dots A_{y_0}. \end{aligned}$$

In particular, $H^s_{x \leftarrow y} = \lim_{n \to \infty} A^n(x)^{-1} A^n(y) = I$ for all $y \in W^s_{loc}(x)$. Analogously, we get the existence of the unstable holonomies.

Chapter 4

Proof of the Main Theorem

In order to motivate the main theorem of this chapter we start proving a characterization of uniform hyperbolicity for a one-step cocycle taking values on $SL(2,\mathbb{R})$.

Proposition 4.1. *Let (T, A) be a one-step $SL(2,\mathbb{R})$-cocycle. Suppose there is a constant $\tau > 0$ and a full probability set $\mathcal{S} \subset \mathcal{R}$ such that*

$$\lambda_+(x) \geq \tau \quad \textit{for every } x \in \mathcal{S}.$$

Then the cocycle (T, A) is uniformly hyperbolic.

Proof. By contradiction, we suppose that for all $\epsilon > 0$ and $n_* \in \mathbb{N}$, exist $n_0 \geq n_*$ and $x \in X$ such that $\|A^{n_0}(x)\| \leq e^{\epsilon n_0}$. Let $x = (\dots, x_{-1}|x_0, x_1, \dots)$ and $p = (\dots, x_{n_0-1}|x_0, x_1, \dots, x_{n_0-1}, \dots) = T^{n_0}p$ a periodic point of order n_0. As the one-step cocycle satisfies

$$A^{n_0}(x) = A_{x_{n_0-1}}A_{x_{n_0-2}}\dots A_{x_0} = A^{n_0}(p),$$

in particular $\|A^{n_0}(p)\| \leq e^{\epsilon n_0}$ and

$$\lambda_+(p) \leq \frac{\log\|A^{n_0}(p)\|}{n_0} = \epsilon.$$

This gives a contradiction since each periodic point p is in every Borel set $\mathcal{S} \subset X$ of full probability and ϵ can be chosen less than τ. ■

Generalizing last proposition, we proceed to prove the following theorem for $SL(2,\mathbb{R})$ fiber-bunched cocycles.

Theorem 4.2. *Let (T, A) be a $SL(2,\mathbb{R})$-cocycle defined over a transitive subshift of finite type or a transitive Anosov diffeomorphism. Suppose the cocycle satisfies the fiber-bunching condition, and there is a constant $\tau > 0$ and a full probability set $\mathcal{S} \subset \mathcal{R}$ such that*

$$\lambda_+(x) \geq \tau \quad \textit{for every } x \in \mathcal{S}.$$

Then the cocycle (T, A) is uniformly hyperbolic.

4.1 Case 1: Subshift of finite type

Let us start by proving Theorem 4.2 for $SL(2,\mathbb{R})$-cocycles over a transitive subshift of finite type.

Proof. Let T be a subshift of finite type and let (T, A) be a fiber bunched $SL(2,\mathbb{R})$-cocycle. Suppose (T, A) is not uniformly hyperbolic. By Proposition 3.2, for all $\epsilon > 0$ and $n_* \in \mathbb{N}$, there exist $n_0 \geq n_*$ and $x = (\dots, x_{-1}|x_0, x_1, \dots) \in X$ such that $\|A^{n_0}(x)\| \leq e^{\epsilon n_0}$. Since Q is irreducible, there is n_1 depending on x_{n_0} and x_0, such that $Q^{n_1}_{x_{n_0}x_0} > 0$. Hence, there is $(c_1, c_2, \dots, c_{n_1-1}) \in \{1, 2, \dots, l\}^{n_1-1}$ such that

$$q_{x_{n_0}c_1} = 1, \quad q_{c_{n_1-1}x_0} = 1, \quad \text{and} \quad q_{c_ic_{i+1}} = 1 \quad \text{for every } i \in \{1, 2, \dots, n_1 - 2\}.$$

Let $p = T^{n_0+n_1}p$ a periodic point of period $n_0 + n_1$, with zeroth coordinate x_0 such that $(p_n)_{n=0}^{n_0+n_1} = (x_0, x_1, \dots, x_{n_0-1}, x_{n_0}, c_1, c_2, \dots, c_{n_1-1})$. Let

$$y = [p, x] = (\dots x_0, x_1, \dots, x_{n_0-1}, x_{n_0}, c_1, c_2, \dots, c_{n_1-1}|x_0, x_1, \dots).$$

By construction $T^{n_0}y \in W^u_{loc}(T^{n_0}p)$ and $p \in W^u_{loc}(y)$, then

$$A^{n_0}(p) = H^u_{T^{n_0}p \leftarrow T^{n_0}y} \cdot A^{n_0}(y) \cdot H^u_{y \leftarrow p}.$$

Analogously, since $T^{n_0}x \in W^s_{loc}(T^{n_0}y)$ and $y \in W^s_{loc}(x)$,

$$\begin{aligned} A^{n_0+n_1}(p) &= A^{n_1}(T^{n_0}p) \cdot A^{n_0}(p) \\ &= A^{n_1}(T^{n_0}p) \cdot H^u_{T^{n_0}p \leftarrow T^{n_0}y} \cdot H^s_{T^{n_0}y \leftarrow T^{n_0}x} \cdot A^{n_0}(x) \cdot H^s_{x \leftarrow y} \cdot H^u_{y \leftarrow p}. \end{aligned}$$

If we take the norm,

$$\|A^{n_0+n_1}(p)\| \leq \|A^{n_1}(T^{n_0}p)\| \cdot \|H^u_{T^{n_0}p \leftarrow T^{n_0}y}\| \cdot \|H^s_{T^{n_0}y \leftarrow T^{n_0}x}\| \cdot \|A^{n_0}(x)\| \cdot \|H^s_{x \leftarrow y}\| \cdot \|H^u_{y \leftarrow p}\|.$$

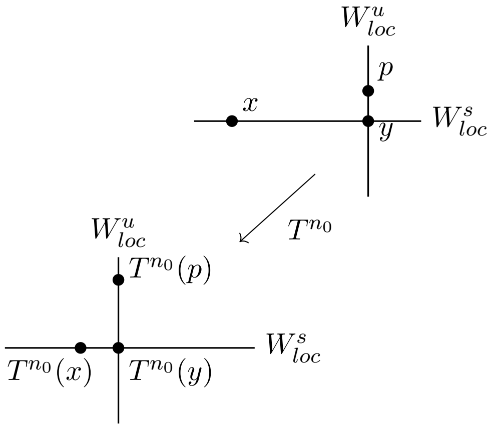

Figure 4.1: Theorem 4.2

It is enough to observe that each term is bounded by a constant C which does not depend on n_0. Note that $\|A^{n_1}(T^{n_0}p)\|$ is bounded as $n_1 < \max_{1\leq i,j\leq n} m_{ij} < \infty$, where m_{ij} are given by Definition 2.2. Hence, by submultiplicativity of the norm,

$$\|A^{n_0+n_1}(p)\| \leq C^5\|A^{n_0}(x)\| \leq C^5 e^{n_0\epsilon}.$$

Therefore

$$\lambda_+(p) \leq 5\frac{\log C}{n_0+n_1} + \frac{n_0\epsilon}{n_0+n_1} \leq 2\epsilon,$$

where the previous inequality follows after choosing n_0 big enough. This gives a contradiction since each periodic point p is in every Borel set $\mathcal{S} \subset X$ of full probability and 2ϵ can be chosen less than τ. ■

4.2 Case 2: Anosov diffeomorphism

4.2.1 Previous lemmas

We proceed to prove Theorem 4.2 for a $SL(2,\mathbb{R})$-cocycle defined over a transitive Anosov diffeomorphism which satisfies the fiber-bunching condition. Firstly, we are going to prove three lemmas that are going to be useful along the proof. The first lemma is a well-known result proved in [BS] page 131. It justifies the transitivity hypothesis in Theorem 4.2.

Lemma 4.3. *Let $T : X \to X$ be an Anosov diffeomorphism of a compact connected manifold X. The following statements are equivalent:*

a) all points in X are non-wandering;

b) every unstable manifold $W^u(x)$ is dense in X;

c) every stable manifold $W^s(x)$ is dense in X;

d) T is topologically transitive.

It is an open problem whether every Anosov diffeomorphism has these four properties.

Proof. Let us assume a), in order to prove b) we will show that every unstable manifold $W^u(x)$ is ϵ-dense for an arbitrary $\epsilon > 0$. In other words, for every $\epsilon > 0$ and $z \in X$, $d(W^u(x), z) < \epsilon$. By Corollary 2.17 the set of periodic points are dense in $NW(T) = X$, so there exists a subset of periodic points $A := \{x_i\}_{i=1}^N$ which form an $\epsilon/4$-net in X. Let $P := \prod_{i=1}^N per(x_i)$ the product of the periods of the points in A. Let us set $\tilde{T} := T^P$, note that the stable and unstable manifolds of $\tilde{T}$ and T are the same.

Claim 4.4. There is a positive integer q such that if $d(W^u_{loc}(y), x_i) < \epsilon/2$ and $d(x_i, x_j) < \epsilon/2$ for some $y \in M$ and some integers $i, j \in \{1, 2, \ldots, N\}$; then $d(\tilde{T}^{nq}(W^u_{loc}(y)), x_i) < \epsilon/2$ and $d(\tilde{T}^{nq}(W^u_{loc}(y)), x_j) < \epsilon/2$ for every $n \in \mathbb{N}$.

Proof. By the local product structure of T there is $z \in W^u_{loc}(y) \bigcap W^s_{loc}(x_i)$. Since $x_i = \tilde{T}(x_i)$ is a fixed point of $\tilde{T}$ then $d(\tilde{T}^m z, x_i) = d(\tilde{T}^m z, \tilde{T}^m x_i) < \epsilon$ for any $m \geq m_1$, where m_1 depends on ϵ but not on z. By the triangular inequality $d(\tilde{T}^m z, x_j) < \epsilon$ for every $m \in \mathbb{N}$, so by the local product structure of T there is $w \in W^u_{loc}(\tilde{T}^m z) \bigcap W^s_{loc}(x_j)$. Since $x_j = \tilde{T}(x_j)$ is a fixed point of $\tilde{T}$, then $d(\tilde{T}^m w, x_j) = d(\tilde{T}^m w, \tilde{T} x_j) < \epsilon$ for any $m \geq m_2$, where m_2 depends on ϵ, but not on w. Finally, the lemma follows by defining $q := m_1 + m_2$. ∎

Since X is compact and connected, any x_i can be connected to any x_j by a chain of no more than N periodic points in A. In other words, for any $x_i \in A$ there is a $x_j \in A \setminus \{x_i\}$ such that $d(x_i, x_j) < \epsilon/2$ and for every $x_i, x_j \in A$ there is a sequence $\{a_i\}_{r=1}^k \subset A$ such that $a_1 = x_i$, $a_k = x_j$ and $d(a_r, a_{r+1}) < \epsilon/2$ for every $r \in \{1, 2, \ldots, k-1\}$ for some positive integer $k \leq N$. By Claim 4.4, we get that $\tilde{T}^{Nq} W^u_{loc}(y)$ is ϵ-dense in X for any $y \in X$. In particular $W^u(x)$ is ϵ-dense for any $x = \tilde{T}^{-Nq}(y) \in X$. Analogously, a) implies c).

Claim 4.5. If every unstable manifold is dense in X, then for every $\epsilon > 0$ there is $R = R(\epsilon) > 0$ such that every ball $W^u_R(x)$ of radius R is ϵ-dense in X.

Proof. By definition for every $x \in X$ the unstable manifold $W^u(x) = \bigcup_{R>0} W^u_R(x)$ is dense, in particular there is $R(x)$ such that $W^u_{R(x)}(x)$ is $\epsilon/2$-dense. By the stable manifold theorem, $W^u(x)$ is continuous so there is $\delta(x) > 0$ such that $W^u_{R(x)}(y)$ is ϵ-dense for every $y \in B(x, \delta(x))$. By the compactness of X there is a finite subcover $\{B(x_i, \delta(x_i))\}_{i=1}^{k_0}$ of X. Clearly $R := \max\{R(x_i) : 1 \leq i \leq k_0\}$ satisfies the lemma. ■

Now, we are going to prove that $b)$ implies $e)$. Let $U, V \subset X$ be two non-empty open sets. Let $x, y \in X$ and $\delta > 0$ such that $W^u_\delta(x) \subset U$ and $B(y, \delta) \subset V$. Let $R = R(\delta)$ given by Claim 4.5, since T expands the unstable manifolds exponentially and uniformly, there is a positive integer N such that $W^u_R(T^n x) \subset T^n(W^u_\delta(x))$ for every $n \geq N$. By Claim 4.5, we get that $T^n(U) \bigcap V \neq \emptyset$ for every $n \geq N$. Hence, T is topologically mixing. The proof of $c)$ implies $e)$ follows analogously. Finally, it is always true that $d)$ implies $a)$. ■

Let $d(\cdot, \cdot)$ be the distance induced by the Riemannian metric on X. Let $d_s(\cdot, \cdot)$ and $d_u(\cdot, \cdot)$ be the induced metrics on $W^s(x)$ and $W^u(x)$ respectively. In addition, the set $W^s_R(x) \subset W^s(x)$ will denote the ball of radius R centered in x with respect to d_s. The definition of $W^u_R(x)$ is analogous. Note that T is a contraction with respect to d_s. More precisely, $d_s(T^n x, T^n y) \leq \lambda^n d_s(x, y)$ for any $x \in X$, $y \in W^s(x)$ and $n \geq 0$. For more details see [BS].

Lemma 4.6. *There is a positive constant R_0 such that for every pair of points $x, y \in X$ we have $W^u_{R_0}(x) \bigcap W^s_{R_0}(y) \neq \emptyset$.*

Proof. Let x an arbitrary point in X. By Lemma 4.3 $W^u(x)$ is dense on X for every $x \in X$. In particular, the global stable manifold $W^s(x)$ intersects the ball $B(z, \delta_1/2)$ centered on z of radius δ_1 for every $z \in X$ where δ_1 is provided by the local product structure of X. Let $x' \in W^s(x) \cap B(z, \delta_1/2)$, by the local product structure $[y, x'] = W^u_{loc}(y) \bigcap W^s_{loc}(x') \in W^u(y) \bigcap W^s(x)$ for every $y \in B(z, \delta_1/2)$. Now, by the compactness of X there is a finite subcover $\{B(z_j, \delta_1)\}_{j=1}^n$ of X by balls of radius δ_1. Note that for every $y \in X$ there exists $\tilde{z} \in \{z_j\}_{j=1}^n$ such that $y \in B(\tilde{z}, \delta_1)$. By the previous argument $W^u_\epsilon(x) \bigcap W^s_{R_s(x)}(x) \neq \emptyset$ for every $x \in X$, where $R_s(x) = \max\{d_s(x, B(z_j, \delta_1)) : 1 \leq j \leq n\} + \epsilon$ and ϵ is the large of the local unstable manifolds.

Analogously, $W^u_{R_u(x)}(x) \bigcap W^s_\epsilon(x) \neq \emptyset$ for every $x \in X$, where $R_u(x) = \max\{d_u(x, B(z_j, \delta_1)) : 1 \leq j \leq n\} + \epsilon$ and ϵ is the large of the local stable manifolds.

Furthermore, $\tilde{R_s} := \max\{R_s(x) : x \in X\} < \infty$ and $\tilde{R_u} := \max\{R_u(x) : x \in X\} < \infty$ since they are the maximum of two continuous functions over a compact space X. Let us define $R_0 := \max\{\tilde{R_s}, \tilde{R_u}\}$. By the previous construction, $W^u_{R_0}(x) \bigcap W^s_{R_0}(y) \neq \emptyset$ for every pair of points $x, y \in X$. ■

Lemma 4.7. *Let $x \in X$ and $\epsilon > 0$. There is a positive integer N independent of x such that for every $n_0 \geq N$ and $z \in W^u_{R_0}(x) \bigcap W^s_{R_0}(T^{n_0}x)$ there is a periodic point $p = T^n p$ such that $d(z,p) < \epsilon$.*

Proof. By Lemma 4.6, $W^u_{R_0}(x) \bigcap W^s_{R_0}(T^{n_0}x) \neq \emptyset$ for every $x \in X$. Let $z \in W^u_{R_0}(x) \bigcap W^s_{R_0}(T^{n_0}x)$. By the stable manifold theorem there is a positive constant λ such that

$$d_s(T^n x, T^n z) \leq \lambda^n d_s(x,z) \leq R_0 \lambda^n < \epsilon,$$

for every n large enough. Hence, there is a positive integer n_1 such that

$$d_s(T^n x, T^n z) < \epsilon \quad \text{for every } x \in X \text{ and } n \geq n_1.$$

Analogously there is a positive integer n_2 such that

$$d_u(T^{-n} z, T^{-n}(T^{n_0}x)) < \epsilon \quad \text{for every } x \in X \text{ and } n \geq n_2.$$

Let $\hat{n} = \max\{n_1, n_2\}$, for $n > 2\hat{n}$ we can consider the periodic ϵ-pseudo-orbit $\{x_k\}_{k=1}^n$ defined by $x_i = T^i z$ if $i \in \{0, 1, \dots \hat{n} - 1\}$, $x_i = T^i x$ if $i \in \{\hat{n}, \hat{n}+1, \dots n - \hat{n} - 1\}$, $x_i = T^{-(n-i)} z$ if $i \in \{n - \hat{n}, n - \hat{n} + 1, \dots n\}$. Graphically,

$$z \mapsto Tz \mapsto \cdots \mapsto T^{\hat{n}-1}z \mapsto T^{\hat{n}}x \mapsto T^{\hat{n}+1}x \mapsto \dots$$

$$\cdots \mapsto T^{n-\hat{n}-1}x \mapsto T^{-\hat{n}}z \mapsto T^{-\hat{n}+1}z \mapsto \cdots \mapsto z.$$

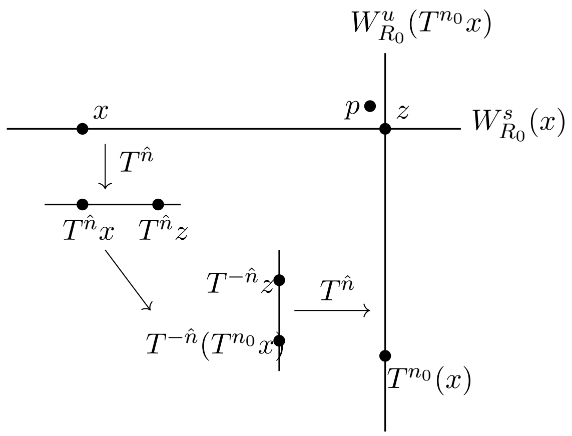

Figure 2: Lemma 4.7

The previous inequalities imply that $d(T(T^{\hat{n}-1}z), T^{\hat{n}}x) < \epsilon$ and $d(T(T^{n-\hat{n}-1}x), T^{-\hat{n}}z) < \epsilon$, hence $\{x_k\}_{k=1}^n$ is a periodic ϵ-pseudo-orbit. By the Anosov closing lemma there is a periodic point $p = T^n p$ such that $d(T^k p, x_k) < C\epsilon$ for every $k \in \{0, 1, \dots, m\}$. In particular $d(z,p) < \epsilon$. Hence, it is enough to consider $N = 2\hat{n} + 4 > 2\hat{n}$. ∎

4.2.2 Proof

Finally, we go on with the proof of Theorem 4.2 for a $SL(2,\mathbb{R})$-cocycle defined over a transitive Anosov diffeomorphism.

Proof. Let T be an Anosov diffeomorphism and let (T,A) be a fiber-bunched $SL(2,\mathbb{R})$-cocycle. Suppose (T,A) is not uniformly hyperbolic. By Proposition 3.2, for all $\epsilon > 0$ and $n_* \in \mathbb{N}$, exist $n_0 \geq n_*$ and $x \in X$ such that $\|A^{n_0}(x)\| \leq e^{\epsilon n_0}$. Along this proof we are going to consider stable manifolds of size R_0, where R_0 comes from Lemma 4.6. We choose $z \in W^u_{R_0}(x) \bigcap W^s_{R_0}(T^{n_0}x)$ which exists by Lemma 4.6. By Lemma 4.7 there is a point p such that $d(z,p) < \delta_1$, where $\delta_1 > 0$ is such that for every $x, y \in X$ the intersection $W^u_{loc}(x) \bigcap W^s_{loc}(y)$ is well defined when $d(x,y) < \delta_1$. Let us define $y = [p,z] \in W^u_{loc}(p) \bigcap W^s_{loc}(z)$. Note the expression

$$H^u_{p \leftarrow T^{n_0}y} \cdot H^s_{T^{n_0}y \leftarrow T^{n_0}x} \cdot A^{n_0}(x) \cdot H^s_{x \leftarrow y} \cdot H^u_{y \leftarrow p}$$

is well defined and equals to $A^{n_0}(p)$. Then

$$\begin{aligned}
\|A^{n_0}(p)\| &\leq \|H^u_{p \leftarrow T^{n_0}y}\| \cdot \|H^s_{T^{n_0}y \leftarrow T^{n_0}x}\| \cdot \|A^{n_0}(x)\| \cdot \|H^s_{x \leftarrow y}\| \cdot \|H^u_{y \leftarrow p}\| \\
&\leq \|H^u_{p \leftarrow y}\| \cdot \|H^u_{y \leftarrow T^{n_0}y}\| \cdot \|H^s_{T^{n_0}y \leftarrow T^{n_0}x}\| \cdot \|A^{n_0}(x)\| \cdot \|H^s_{x \leftarrow y}\| \cdot \|H^u_{y \leftarrow p}\|.
\end{aligned}$$

To conclude the proof it is enough to note that each term is bounded by a constant C depending on the size of the unstable and stable manifolds under consideration. The only term which is not clearly bounded is $\|H^u_{y \leftarrow T^{n_0}y}\|$. In order to bound this term we state the following lemma which follows directly from the continuity of the stable manifolds.

Lemma 4.8. *Let $x, y \in X$. For all $R_0 > 0$ there exists $\epsilon_1 < \epsilon_0$ such that if $y = W_{R_0}(x)$ and $y' \in W^s_{\epsilon_1}$ then there is a unique point $x' \in X$ such that $W^u_{R_0+2\epsilon_0}(y') \bigcap W^s_{\epsilon_0}(x) = \{x'\}$.*

Applying the previous lemma to $y := z$, $x := T^{n_0 x}$, $y' := y$ and $x' := T^{n_0}y$ we get a bound for $\|H^u_{y \leftarrow T^{n_0}y}\|$ depending on the size of the unstable and stable manifolds under consideration.

Hence, by submultiplicativity of the norm,

$$\|A^{n_0}(p)\| \leq C^5 \|A^{n_0}(x)\| \leq C^5 e^{n_0 \epsilon}.$$

Therefore

$$\lambda_+(p) \leq 5\frac{\log C}{n_0} + \epsilon \leq 2\epsilon,$$

where the previous inequality follows after choosing n_0 big enough. This gives a contradiction since each periodic point p is in every Borel set $\mathcal{S} \subset X$ of full probability and 2ϵ can be chosen less than τ. ∎

Remark. In particular, we could have changed the transitivity hypothesis in the Anosov case for any of the other three equivalent properties in Lemma 4.3.

Remark. More precisely, we showed that a $SL(2,\mathbb{R})$-cocycle over a transitive subshift of finite type or a transitive Anosov diffeomorphism is uniformly hyperbolic if and only if it has uniform gap for every invariant probability measure supported on a periodic orbit. Nevertheless, this is not surprising since B. Kalinin proved in [K] that the Lyapunov exponents of a linear cocycle (T, A) can be arbitrarily approximated by Lyapunov exponents of measures supported on periodic orbits. As a result, the cocycle (T, A) has uniform gap for every T-invariant probability measure if and only if it has uniform gap for every T-invariant probability measure supported on a periodic orbit.

Remark. The previous proof works identically for a cocycle over a hyperbolic homeomorphism. In other words, it is not necessary to consider a cocycle over an Anosov diffeomorphism to get the result. See [Sak] for more details on hyperbolic homeomorphisms.

Chapter 5

A Counterexample

In this chapter we show the fiber-bunching condition is necessary for the validity of Theorem 1.5. Before the proof of Theorem 1.6, we are going to illustrate the usefulness of some invariant families of cones in order to prove that a diffeomorphism is Anosov and a cocycle is uniformly hyperbolic.

5.1 Invariant cones

During this section we present two results which show the usefulness of some invariant families of cones in order to prove that a diffeomorphism is Anosov and a cocycle is uniformly hyperbolic. This perspective will be useful during the construction on the example in the next section.

Firstly, we show a sufficient condition to prove that $T : X \to X$ is an Anosov diffeomorphism. It assumes the existence of two linear cones $K^u_\alpha(x) := \{v \in T_xX : \|v^s\| \leq \alpha\|v^u\|\}$ and $K^s_\alpha(x) := \{v \in T_xX : \|v^u\| \leq \alpha\|v^s\|\}$ where $v^u \in \tilde{E}^u_x$ and $v^s \in \tilde{E}^s_x$ for some splitting $\tilde{E}^s_x \oplus \tilde{E}^u_x = T_xX$ of the tangent bundle.

Proposition 5.1. *Let X be a smooth compact manifold and $T : X \to X$ be a diffeomorphism. Suppose there is $\alpha > 0$ such that for every $x \in X$ there are continuous subspaces $\tilde{E}^s_x$ and $\tilde{E}^u_x$ such that $\tilde{E}^s_x \oplus \tilde{E}^u_x = T_xX$, and the α-cones $K^s_\alpha(x)$ and $K^u_\alpha(x)$ determined by the subspaces satisfy*

$$DT_xK^u_\alpha(x) \subset K^u_\alpha(Tx) \quad \textit{and} \quad DT^{-1}_{Tx}K^s_\alpha(Tx) \subset K^s_\alpha(x)$$

and

$$\|DT^{-1}_x v\| \leq \lambda\|v\| \quad \textit{for every } v \in K^u_\alpha(x),$$
$$\|DT_x v\| \leq \lambda\|v\| \quad \textit{for every } v \in K^s_\alpha(x).$$

where $\lambda \in (0,1)$ is constant. Then the mapping T is an Anosov diffeomorphism.

Proof. Let us define

$$E_x^u = \bigcap_{n=0}^{\infty} DT^n_{T^{-n}x} K^u_\alpha(T^{-n}(x)) \quad \text{and} \quad E_x^s = \bigcap_{n=0}^{\infty} DT^{-n}_{T^n x} K^s_\alpha(T^n(x)).$$

Note that E_x^s has nonzero vectors since $DT^{-(n+1)}_{T^{n+1}x} K^s_\alpha(T^{n+1}(x)) \subset DT^{-n}_{T^n x} K^s_\alpha(T^n(x))$ for every $n \in \mathbb{N}$. Analogously, E_x^u has nonzero vectors since $DT^{n+1}_{T^{-(n+1)}x} K^u_\alpha(T^{-(n+1)}(x)) \subset DT^n_{T^{-n}x} K^u_\alpha(T^{-n}(x))$. The same argument proves the invariance of E_x^s and E_x^u. Furthermore

$$\|DT_x^n v^s\| \leq \lambda^n \|v^s\| \quad \text{and} \quad \|DT_x^{-n} v^u\| \leq \lambda^n \|v^u\|,$$

for every $x \in X$, $v^s \in E_x^s$, $v^u \in E_x^u$ and $n \geq 1$. Since $DT^{n-1}_{T^{n-1}(x)} v^s \in K^s_\alpha(T^{n-1}(x)), \ldots,$ $DT_{T(x)} v^s \in K^s_\alpha(T(x))$ and $v^s \in K^s_\alpha(x)$,

$$\begin{aligned}
\|DT_x^n v^s\| &= \|DT_{T^{n-1}x} \cdots DT_{Tx} DT_x v^s\| \\
&\leq \lambda \|DT_{T^{n-2}x} \cdots DT_{Tx} DT_x v^s\| \\
&\leq \lambda^2 \|DT_{T^{n-3}x} \cdots DT_{Tx} DT_x v^s\| \\
&\leq \cdots \leq \lambda^n \|v^s\|,
\end{aligned}$$

for every $x \in X$, $v^u \in E_x^u$ and $n \geq 1$. Analogously $\|DT_x^{-n} v^u\| \leq \lambda^n \|v^u\|$ for every $x \in X$, $v^u \in E_x^u$ and $n \geq 1$. ■

Now, we present a characterization of uniform hyperbolicity for a one-step cocycle in terms of a familiy of cones M. This result is due to A. Avila, J.Bochi and J.-C. Yoccoz in [ABY].

Theorem 5.2. *A one-step cocycle (T, A) defined by the left shift map $T : \{1, 2, \ldots, n\}^{\mathbb{Z}} \to \{1, 2, \ldots, n\}^{\mathbb{Z}}$ and $A(x) = A_{x_0}$ for some set of matrices $\{A_1, A_2, \ldots, A_n\} \subset GL(d, \mathbb{R})$ is uniformly hyperbolic if and only if there exists a nonempty homogeneous open subset $M \subset \mathbb{R}^2$ with $\overline{M} \neq \mathbb{R}^2$ such that $\overline{A_\alpha M} \subset Int(M)$ for every $\alpha \in \{1, 2, \ldots, n\}$. We can take M with finitely many connected components and those components with disjoint closures.*

See [ABY] in order to see a proof of Theorem 5.2. Now, we proceed to construct the example with the properties mentioned in Theorem 1.6.

5.2 Construction of the example

In the following, we are going to exhibit a cocycle which has uniform gap between the Lyapunov exponents in a set of full probability $\mathcal{S}$, however it is not uniformly hyperbolic. In particular, it cannot satisfy the fiber-bunching condition. This example shows that the fiber-bunching condition is necessary in Theorem 4.2. See [CLR] and [G] for more complex constructions of cocycles with similar properties.

Let $X = \{0,1\}^{\mathbb{Z}}$ and $T : X \to X$ the left shift map. We consider a cocycle $A : X \to SL(2,\mathbb{R})$ defined by

$$A(x) := \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} R_{\theta(x)},$$

where the function $R_{\theta(x)}$ is the rotation of angle $\theta(x)$. Let $V := \{x \in X : x_0 = 1\}$ be a neighbourhood of $q := (\dots, 0, 0|1, 0, 0, \dots)$. Let us define θ as

$$\theta(x) := \begin{cases} \pi/2 & \text{if } x = q, \\ \pi/2 - 2^{-k(x)/8} & \text{if } x \in V \setminus \{q\} \text{ and } k(x) > k_0, \\ 0 & \text{if } x = \vec{0} \text{ or } k(x) \leq k_0, \end{cases}$$

where $k(x) := \min\{|n|; n \neq 0, x_n = 1\}$ and k_0 is a positive integer which will be defined later. Note that $\theta(x) \in [0, \pi/2]$ for every $x \in X$. Also, we observe that when x tends to q, $k(x)$ tends to infinity, hence $\theta(x)$ tends to $\pi/2$. In particular, A is continuous as required. More precisely, we proceed to prove the following theorem.

Theorem 5.3. *The $SL(2,\mathbb{R})$-cocycle (T, A) defined above has the following properties:*

1. *The cocycle (T, A) is not uniformly hyperbolic.*
2. *There is a set of full probability $\mathcal{S}$, such that $\lambda_+(x) \geq \log 2/2 > 0$ for every $x \in \mathcal{S}$.*

Claim 5.4. The $SL(2,\mathbb{R})$-cocycle (T, A) is not uniformly hyperbolic.

The proof of this claim is identical to the proof given in the Example 3.1.1.

Proof. Since

$$\lim_{n \to \pm\infty} T^n q = \vec{0} = (\dots, 0|0, 0, \dots) = T(\vec{0}),$$

by definition q is a homoclinic point for the fixed point $\vec{0}$. In addition $R_{\theta(q)} = R_{\pi/2}$, therefore the cocycle cannot be uniformly hyperbolic. Let us suppose that (T, A) is uniformly hyperbolic, by the invariance of E^s_x

$$E^s_{T^n q} = A^{2n}(T^{-n}q)E^s_{T^{-n}q} = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^n R_{\pi/2} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^n E^s_{T^{-n}q}$$

as q is the only point of V in the orbit of q. Besides, by the continuity of E^s_x, we have $E^s_{T^n q} \approx E^s_{\vec{0}} = \{x = 0\}$ for n big enough. Hence, we would have that $E^s_{T^{-n}q} \approx \{y = 0\}$. However, by the continuity of E^s_x, we have $E^s_{T^{-n}q} \approx E^s_{\vec{0}} = \{x = 0\}$, a contradiction. ■

Claim 5.5. There is a set of full probability $\mathcal{S}$, such that $\lambda_+(x) \geq \log 2/2 > 0$ for every $x \in \mathcal{S}$.

In the following, we are going deal with a linear cocycle induced by (T, A) and the neighbourhood V. Let $V_0 := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} T^{-n}(V) \bigcap V$ the set of points in V which return infinitely many times to V. Let $T_V : V_0 \to V_0$ be the first return map defined by

$$T_V(x) := T^{N_V(x)}(x), \quad \text{where } N_V(x) := \inf\{n \geq 1 : T^n(x) \in V_0\}.$$

Let $A_V : V \to SL(2, \mathbb{R})$ be the function defined by $A_V(x) := A^{N_V(x)}(x)$. Note that (T_V, A_V) is a linear cocycle, however V_0 is not compact. We proceed to prove the key lemma in order to show the gap between the Lyapunov exponents.

Remark. During the proof of the following lemma, we are going to use repeatedly that $k(x) \leq N_V(x)$ and $k(T_V(x)) \leq N_V(x)$ for every $x \in X$.

We say a set $\mathcal{C} \subset \mathbb{R}^2$ is a *cone* if it is a homogeneous space between two transverse one-dimensional spaces. In the following lemma, we prove the existence of a family of invariant cones $\mathcal{C}(x)$ for each $x \in V_0$.

Lemma 5.6. *For every $x \in V_0$ there is a cone $\mathcal{C}(x) \subset \mathbb{R}^2$ such that*

$$A_V(x)\mathcal{C}(x) \subset \mathcal{C}(T_V x).$$

Moreover, for every unit vector $v \in \mathcal{C}(x)$, we have $\|A_V(x)v\| \geq 2^{N_V(x)/2}$.

Proof. By definition

$$A_V(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^{N_V(x)} R_{\theta(x)}.$$

Let us define $\beta(x) := 2^{-k(x)/2+1/4}$. Note that

$$0 < \frac{\pi}{2} - \theta(x) - \beta(x) < \frac{\pi}{2} - \theta(x) + \beta(x) < \frac{\pi}{2}.$$

In facto, by the definition of $\beta(x)$, $\theta(x) > \beta(x)$ for every $x \in V_0$. Besides, $\pi/2 > \theta(x) + \beta(x)$ is equivalent to

$$\frac{\pi}{2} > 2^{-k(x)/8} + 2^{-k(x)/2+1/4}.$$

If $k(x)$ is big enough $2^{-k(x)/8} < 0.3$, so

$$\theta(x) + \beta(x) = 2^{-k(x)/8} + 2^{-k(x)/2+1/4} < 0.3 + 2^{1/4} < \frac{\pi}{2}.$$

Due to last condition, it makes sense define the cone

$$\mathcal{C}(x) = \mathbb{R}^2 \setminus \left\{(x, y) \in \mathbb{R}^2 : \tan\left(\frac{\pi}{2} - \theta(x) - \beta(x)\right) \leq \frac{y}{x} \leq \tan\left(\frac{\pi}{2} - \theta(x) + \beta(x)\right)\right\}$$

showed on Figure 2. We proceed to prove that the cone $\mathcal{C}(x)$ satisfies the lemma.

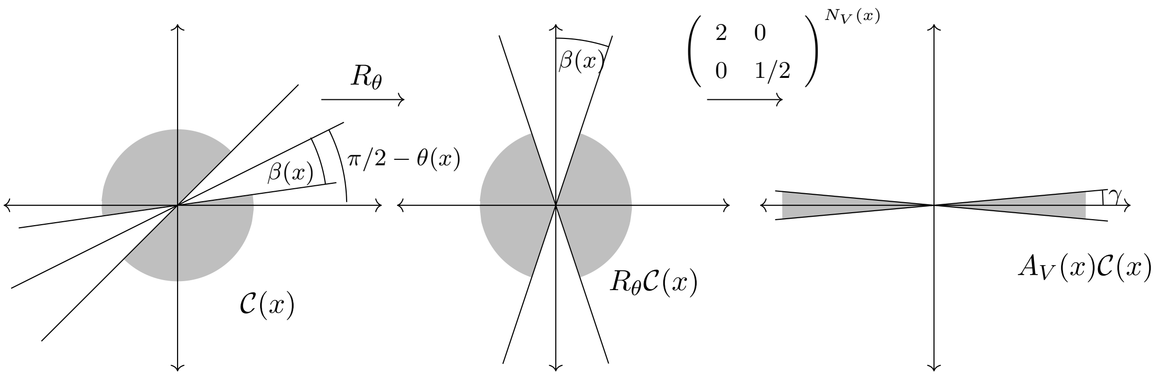

Figure 5.2: Lemma 5.6

Let v be a unit vector in $\mathcal{C}(x)$. If $k(x)$ is big enough, then $\sin\beta(x) > 2^{-1/8}\beta(x)$. Hence

$$\|A_V(x)v\| \geq 2^{N_V(x)} \sin\beta(x) \geq 2^{N_V(x)-k(x)/2+1/8} \geq 2^{N_V(x)/2+1/8} \geq 2^{N_V(x)/2}.$$

Let γ be the greatest angle in the first quadrant between a vector in $A_V(x)\mathcal{C}(x)$ and the x axis as we show in the third image of Figure 5.2. Consequentially, it is enough to prove that

$$\gamma(x) < \frac{\pi}{2} - \theta(T_V x) - \beta(T_V x) \quad \text{for every } x \in V_0,$$

in order to get the invariant condition. We can assume the vectors

$$(\cos\gamma(x), \sin\gamma(x)) \quad \text{and} \quad (2^{N_V(x)} \sin\beta(x), 2^{-N_V(x)} \cos\beta(x)),$$

are linearly dependent. If $k(x)$ is big enough, $\cot\beta(x) \leq 2^{1/8}\beta(x)^{-1} = 2^{k(x)/2-1/8}$. Hence

$$\tan\gamma(x) = 2^{-2N_V(x)}\cot\beta(x) \leq 2^{-2N_V(x)+k(x)/2-1/8} \leq 2^{-3N_V(x)/2-1/8},$$

which tends to zero when $k(x)$ tends to infinite. As $\gamma(x) \in (0,\pi/2)$, we conclude that $\gamma(x)$ tends to zero. By the previous calculation,

$$\gamma(x) \leq 2^{1/16}\tan\gamma(x) \leq 2^{-3N_V(x)/2-1/16} \leq 2^{-3k(T_Vx)/2-1/16},$$

if $k(x)$ is big enough. Finally we show that

$$\frac{\pi}{2} - \theta(T_Vx) - \beta(T_Vx) = 2^{-k(T_Vx)/8} - 2^{-k(T_Vx)/2+1/4}$$

$$= 2^{-k(T_Vx)/2}(2^{3k(T_Vx)/8} - 2^{1/4}) \geq 2^{-k(T_Vx)/2} \geq 2^{-3k(T_Vx)/2-1/16}$$

The previous two inequality series prove that $A_V(x)\mathcal{C}(x) \subset \mathcal{C}(T_Vx)$ when $k(x) > k_0$ for some big enough positive integer k_0. ■

Finally, let μ be an ergodic T-invariant measure. If $\mu(V) = 0$, then $\theta(x) = 0$ and consequently $\lambda_+(x) = \log 2$ μ-almost everywhere. Otherwise, by Poincaré recurrence theorem $\mu(V) = \mu(V_0)$ for every T-invariant probability measure μ. Let us define $\mathcal{C}_\infty(x) := \bigcap_{n=0}^{\infty} A_V^n(T_V^{-n}x)\mathcal{C}(T_V^{-n}x)$. Note that $\mathcal{C}_\infty(x)$ has nonzero vectors since $A_V^{n+1}(T_V^{-(n+1)}x)\mathcal{C}(T_V^{-(n+1)}x) \subset A_V^n(T_V^{-n}x)\mathcal{C}(T_V^{-n}x)$ for every $n \in \mathbb{N}$. By Lemma 5.6

$$\|A_V^n(x)v\| \geq 2^{N_V(T_V^{n-1}x)}\|A_V^{n-1}(x)v\| \geq 2^{N_V(T_V^{n-1}x)+N_V(T_V^{n-2}x)+\cdots+N_V(x))/2}\|v\|,$$

for every $x \in V_0$ and $v \in \mathcal{C}_\infty(x)$. Hence, there is a sequence $j_1 < j_2 < \cdots < j_n = N_V(T_V^{n-1}x) + N_V(T_V^{n-2}x) + \cdots + N_V(x)$ such that $\|A^{j_n}(x)\cdot v\| \geq 2^{j_n/2}\|v\|$ for every $v \in \mathcal{C}_\infty$ and $n \in \mathbb{N}$. Consequently, by Oseledets's theorem

$$\lambda_+(x) = \lim_{n\to\infty}\frac{\log\|A_V^n(x)v\|}{n} = \lim_{n\to\infty}\frac{\log\|A_V^{j_n}(x)v\|}{j_n} \geq \frac{\log 2}{2}$$

for every $x \in V_0$ and $v \in \mathcal{C}_\infty(x)$. It proves the gap between the Lyapunov exponents of the $SL(2,\mathbb{R})$-cocycle (T, A) for an arbitrary ergodic measure μ. Furthermore, by the ergodic decomposition theorem we get that $\lambda_+(x) \geq \log 2/2$ for every $x \in V_0$ and every T-invariant measure μ. Note that when $k(x) \leq k_0$ the cocycle does not have rotations, and the Lyapunov exponent λ_+ is equal to $\log 2$. As a result

$$\lambda_+(x) \geq \frac{\log 2}{2} \quad \text{for every } x \in \mathcal{S},$$

where $\mathcal{S} := [\bigcup_{n\in\mathbb{Z}} T^n(X \setminus V) \bigcup V_0] \bigcap \mathcal{R}$ is a set of full probability.

Remark. The cocycle is $1/8$-Hölder continuous as $2^{-k(x)} = d(x,q)$ and $\theta(x) = \pi/2 - 2^{-k(x)/8}$ if $x \in V \setminus \{q\}$ and $k(x) > k_0$. In fact

$$|\theta(x) - \theta(y)| = |d(x,q)^{1/8} - d(y,q)^{1/8}| \leq d(x,y)^{1/8},$$

consequently $\sin\theta(x)$ and $\cos\theta(x)$ are $1/8$-Hölder continuous. Hence $A(x)$ is $1/8$-Hölder continuous. We notice directly that the cocycle does not satisfy the fiber-bunching condition. Generally, a $SL(2,\mathbb{R})$-cocycle which is α-Hölder satisfies the fiber-bunching condition if and only if $\|A(x)\| < 2^{\alpha/2}$. However, last example satisfies $\|A(x)\| = 2$ and $\alpha = 1/8$, so the cocycle does not satisfy the fiber-bunching condition.

Remark. In the previous construction we considered a particular choice for the neighbourhood V of the homoclinic point q. It was useful in order to prove Lemma 5.6, nevertheless one should be able to construct similar examples for others neighbourhoods of q. More generally, one should be able to construct similar examples where T is an Anosov diffeomorphism.

Chapter 6

Final Remarks

In this section we are going to state some natural questions which are motivated by Theorem 4.2 and Theorem 5.3. We start by defining a well-known concept called *dominated splitting*, for more details see [Sam]. In the following (T, A) will denote a $GL(d, \mathbb{R})$-cocycle. In addition, $\sigma_1(M) \geq \cdots \geq \sigma_d(M)$ will be the singular values of a matrix M and $\mathfrak{m}(M) = \inf_{\|v\|=1} \|Mv\|$ will be the co-norm of a matrix M. Note that $\sigma_1(M) = \|M\|$ and $\mathfrak{m}(M) = \|M^{-1}\|^{-1} = \sigma_d(M)$ for every $M \in GL(d, \mathbb{R})$. We proceed with the definition of a dominated splitting.

Definition 6.1. We say that A admits a *dominated splitting of index i* if there is a A-invariant splitting $V = E \oplus F$ where $\dim(E) = i$ and there are constants $C > 0$ and $0 < \tau < 1$ such that

$$\frac{\|A^n(x)|F_x\|}{m(A^n(x)|E_x)} < C\tau^n \quad \text{for every } x \in X \text{ and every } n \geq 0.$$

Remark. Note that for $SL(2, \mathbb{R})$-cocycles, last definition is equivalent to uniform hyperbolicity.

The following theorem proved by J. Bochi and N. Gourmelon in [BG] generalizes Proposition 3.2 to higher dimensions.

Theorem 6.2. *The following assertions about a linear cocycle are equivalent*

a) There is a dominated splitting of index i.

b) There exist $C > 0$ and $\tau < 1$ such that $\dfrac{\sigma_{i+1}(A^n(x))}{\sigma_i(A^n(x))} < C\tau^n$ for all $x \in X$ and $n \geq 0$.

Furthermore, a way to define the intermediate Lyapunov exponents is through singular values. More precisely, we have

$$\lim_{n\to\infty} \frac{1}{n} \log(\sigma_i(A^n(x))) = \lambda_i(x)$$

for every $x \in \mathcal{R}$, where $\mathcal{R}$ is a full probability set. In particular, since $\sigma_1(M) = \|M\|$ and $\|M^{-1}\|^{-1} = \sigma_d(M)$ for every $M \in GL(d,\mathbb{R})$, then $\lambda_+(x) = \lambda_1(x)$ and $\lambda_-(x) = \lambda_d(x)$ for every $x \in \mathcal{R}$. Hence, the existence of a dominated splitting of index i implies the uniform gap between $\lambda_i(x)$ and $\lambda_{i+1}(x)$. By Theorem 6.2,

$$\lambda_i(x) - \lambda_{i+1}(x) \geq \log(\tau^{-1}) \quad \text{for all } x \in \mathcal{S}.$$

Now, we state a direct consequence of the proof of Theorem 4.2 and Theorem 6.2.

Corollary 6.3. *Let (T, A) be a $GL(2,\mathbb{R})$-cocycle defined over a transitive Anosov diffeomorphism, which satisfies the fiber-bunching condition. Then, if there is a constant $\tau > 0$ and a full probability set $\mathcal{S} \subset \mathcal{R}$ such that*

$$\lambda_+(x) - \lambda_-(x) \geq \tau \quad \textit{for all } x \in \mathcal{S},$$

the cocycle (T, A) admits a dominated splitting.

In fact,

$$\frac{\sigma_1(L)}{\sigma_2(L)} = \frac{\sigma_1(L)}{|\det L|/\sigma_1(L)} = \frac{\sigma_1(L)^2}{|\det L|} \geq \frac{\lambda_1(L)^2}{|\det L|} = \frac{\lambda_1(L)}{\lambda_2(L)},$$

which justifies the previous corollary. Naturally, Theorem 6.2 suggests the following question for fiber-buched $GL(d,\mathbb{R})$-cocycles.

Question 6.4. Let (T, A) be a $GL(d,\mathbb{R})$-cocycle defined over a transitive subshift of finite type or a transitive Anosov diffeomorphism. Let us suppose the cocycle (T, A) satisfies the fiber-bunching condition. If there is a constant $\tau > 0$ and a full probability set $\mathcal{S} \subset \mathcal{R}$ such that

$$\lambda_i(x) - \lambda_{i+1}(x) \geq \tau \quad \text{for all } x \in \mathcal{S}.$$

Does the cocycle (T, A) have a dominated splitting of index i?

Finally, it would be interesting to prove the existence of $SL(2,\mathbb{R})$-cocycles with the properties of the example given in Chapter 5 which almost satisfy the fiber-bunching inequality. More precisely, we would like to prove the following.

Question 6.5. Let T be the left-shift map $T : \{1, 2, \ldots l\}^{\mathbb{Z}} \to \{1, 2, \ldots l\}^{\mathbb{Z}}$ defined by $T((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. Let us consider the metric as before in Section 2.1. Let $c > 1$ be an arbitrary positive constant. Is there an α-Hölder $SL(2,\mathbb{R})$-cocycle (T, A) which is not uniformly hyperbolic, but there is a constant $\epsilon > 0$ such that $\lambda_+(x) > \epsilon$ for every point x in a set $\mathcal{S}$ of full probability, and also $\|A(x)\|^2 \cdot 2^{-\alpha} < c$?

Remark. If c were less than 1, this would mean that the cocycle (T, A) is fiber-bunched. Our construction works for any positive constant α less than 1 and $\|A(x)\| = 2$ for every $x \in X$. Hence, it does not satisfy the requirements of Question 6.5.

Bibliography

[AB1] Artur Avila and Jairo Bochi. On the subadditive ergodic theorem. Preprint, 2009. Available at www.mat.uc.cl/ jairo.bochi/docs/kingbirk.pdf.

[AB2] A. Avila, J. Bochi. Trieste lecture notes on Lyapunov exponents. Preprint, 2008. Available at http://www.mat.uc.cl/ jairo.bochi/docs/trieste.pdf.

[AB3] A. Avila and J. Bochi. A formula with some applications to the theory of Lyapunov exponents. Israel Journal of Mathematics 131, 2002, 125-137.

[ABY] A. Avila, J. Bochi, J.-C. Yoccoz. Uniformly hyperbolic finite-valued $SL(2,\mathbb{R})$-cocycles. Comment. Math. Helv. 85, 2010, 813-884.

[BBB] Lucas Backes, Aaron Brown, Clark Butler. Continuity of Lyapunov exponents for cocycles with invariant holonomies. Preprint arXiv:1507.08978.

[BG] J. Bochi, N. Gourmelon. Some characterizations of domination. Math. Z. 263, 2009, no. 1, 221-231.

[BGMV] C. Bonatti, X. Gómez-Mont, M. Viana. Généricité d'exposants de Lyapunov non-nuls pour des produits déterministes de matrices. Ann. Inst. H. Poincaré Anal. Non Linéaire 20, 2003, p. 579-624.

[BS] M. Brin, G. Stuck. Introduction to Dynamical Systems. Cambridge University Press, 2002.

[C] Y. Cao. Non-zero Lyapunov exponents and uniform hyperbolicity. Nonlinearity 16, 2003, 1473-1479

[CLR] Y.Cao, S Luzzatto, I. Rios. Some non-hyperbolic systems with strictly non-zero Lyapunov exponents for all invariant measures: horseshoes with internal tangencies. Discrete Contin. Dyn. Syst. 15, 2006, 61-71.

[F] S. Filip. Notes on the Multiplicative Ergodic Theorem. Preprint arXiv:1710.10694.

[G] A. Gogolev. Diffeomorphisms Hölder conjugate to Anosov diffeomorphisms. Ergodic Theory Dynam. Systems 30, 2010, no. 2, 441-456

[H] M. W. Hirsch, Differential topology. Graduate Texts in Mathematics, Springer-Verlag, Corrected 6th printing, 1997.

[He] M. R. Herman. Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. Comment. Math. Helvetici, 58, pp. 453-502, 1983.

[K] B. Kalinin. Livsic theorem for matrix cocycles. Ann. of Math. 173, 2011, no. 2, 1025-1042.

[KS] B. Kalinin, V. Sadovskaya. Cocycles with one exponent over partially hyperbolic systems. Geometriae Dedicata, Vol. 167, Issue 1, 2013, 167-188.

[KH] A. Katok, B. Hasselblatt. Introduction to the Modern Theory of Dynamical Systems. Encycl. Math. Appl. 54, Cambridge University Press, 1995.

[P] M. Pollicott. Lectures on Ergodic Theory and Pesin Theory on Compact Manifolds. Cambridge University Press, 1993.

[Sak] K. Sakai. Shadowing properties of L-hyperbolic homeomorphisms. Topology and its Appl. 112, 2001, 229-243.

[Sam] M. Sambarino. A (short) survey on dominated splittings. Mathematical Congress of the Americas, 149-183. Contemp. Math. 656, Amer. Math. Soc., 2016.

[Sh] M. Shub. Global stability of dynamical systems. Springer Verlag, 1987.

[V] M. Viana. Lectures on Lyapunov Exponents. Cambridge University Press, 2014.

[Y] J.-C. Yoccoz. Some questions and remarks about $SL(2,\mathbb{R})$ cocycles. In Modern dynamical systems and applications, Cambridge University Press, 447-458.

[Z] Z. Zhang. Uniform hyperbolicity and its applications to spectral analysis of 1D discrete Schrödinger operators. Preprint arXiv:1305.4226v3.