Some old and new examples in ergodic optimization

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Optimization of ergodic averages

$T\colon X \to X$ a continuous transformation of a compact metric space.

$\phi\colon X \to \mathbb{R}$ a continuous function, $S_n\phi := \sum_{j=0}^{n-1} \phi \circ T^j =$ Birkhoff sum

Definition

The *ergodic maximum* of ϕ wrt T is:

$$\beta(\phi \mid T) := \lim_{n\to\infty} \frac{1}{n} \max_{x \in X} (S_n\phi)(x)\,,$$

where $\lim = \inf$ exists by subadditivity (Fekete lemma).

First question: do there exist points x whose orbits are "optimal" in the sense that

$$\lim_{n\to\infty} \frac{(S_n\phi)(x)}{n} = \beta(\phi \mid T) \quad ?$$

Alternative formulation: maximizing measures

$\mathcal{M}_T :=$ (compact convex) set of T-invariant probability measures.

Proposition ($\leq$ Jean-Pierre Conze, Yves Guivarc'h, $\sim$ 1993)

$$\beta(\phi \mid T) = \sup_{\mu \in \mathcal{M}_T} \int_X \phi \, d\mu \, .$$

There is always at least one ergodic measure attaining the sup.

Definition

Maximizing measures are those that attain the sup above.

Problem

How to concretely find maximizing measures (and in particular compute β)? What are their typical properties?

Typically unique and periodic maximization

Theorem (Thierry Bousch, 2000)

*If T is the doubling map on the circle $\mathbb{R}/\mathbb{Z}$ and $\phi_{a,b}(x) = a\cos 2\pi x + b\sin 2\pi x$, then there exists an **open** set $G \subseteq \mathbb{R}^2$ of **full Lebesgue measure** such that if $(a, b) \in G$, then the maximizing measure for $\phi_{a,b}$ is unique and supported on a periodic orbit.*

Theorem (Gonzalo Contreras, 2016)

*If T is an expanding map (e.g. a one-sided shift) and ϕ is a **generic** Lipschitz function, then the maximizing measure is unique and supported on a periodic orbit.*

Theorem (Rui Gao, Weixiao Shen, Ruiqin Zhang, preprint 2025)

*If T is an analytic expanding circle map and ϕ is a **typical (in a probabilistic sense)** C^r function, $r = 1, 2, \ldots, \infty, \omega$, then the maximizing measure is unique and supported on a periodic orbit.*



Bousch Contreras Shen

Maximizing measures tend to have special structure

Theorem (Rui Gao, Weixiao Shen, 2024)

If T is an analytic expanding circle map and ϕ is a real analytic function which is not a coboundary, then all maximizing measures have zero entropy.

In the classical example $T(x) = 2x \bmod 1$, $\phi(x) = a\cos 2\pi x + b\sin 2\pi x$, Bousch actually showed that (unless $\phi \equiv 0$), the maximizing measure is always unique **Sturmian**[1] – the measures of lowest possible complexity.

The recent work of Gao–Shen–Zhang relies on identifying a "Sturmian-like" structure.

[1] identify $[0,1]$ with $\{0,1\}^{\mathbb{N}}$ using binary expansions

Sturmian measures

Definition (Marston Morse, Gustav Hedlund, 1940)

Let μ be a shift-invariant measure on $\{0,1\}^{\mathbb{N}}$. We say that μ is *Sturmian* if, for any $n \geq 1$, at most $n+1$ cylinders of "length" n have positive measure.

Example

The measure supported on the orbit of $(0011)^{\infty}$ is **not** Sturmian, since it intersects the four cylinders [00], [01], [10], [11].

Theorem

A Sturmian measure μ is completely determined by the parameter $\gamma := \mu([1])$. The support of $\mu = \mu_\gamma$ consists of the itineraries of the irrational rotation $x \mapsto x + \gamma \bmod 1$ with respect to the partition $I_0 = [\gamma, 1)$, $I_1 = [0, \gamma)$. This shift-invariant set is uniquely ergodic.

Subadditive ergodic optimization

$T: X \to X$ still a continuous transformation of a metric space.

$\phi_n: X \to \mathbb{R}$ a **subadditive** sequence of continuous functions.

Proposition (many authors)

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} \phi_n(x) = \sup_{\mu\in\mathcal{M}_T} \int_X \underbrace{\lim_{n\to\infty} \frac{\phi_n(x)}{n}}_{Kingman} \, d\mu(x)$$

Similarly to the above, let $\beta((\phi_n) \mid T)$ denote the sup above, and call *maximizing measure* any μ that attains the sup above.

Optimization of Lyapunov exponents

One of our favorite subadditive sequences:

$$\phi_n(x) = \log \|A(T^{n-1}x) \cdots A(x)\|$$

where $A\colon X \to \mathrm{Mat}(d \times d)$. (Recall Matheus' talk.)

The corresponding ergodic maximum $\beta((\phi_n) \mid T)$ is the **maximal Lyapunov exponent** of the linear cocycle (T, A).

$$\beta = \sup_{\mu \in \mathcal{M}} \lambda_1(T, A, \mu)\,.$$

Joint spectral radius

$T =$ shift on m symbols acting on $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$.

$A\colon \Omega \to \mathrm{Mat}(d \times d)$ such that $A(\omega) = A_{\omega_0}$.

Then (T, A) is called the *one-step coccyle* induced by the tuple of matrices $(A_1, \ldots, A_m)$.

If β is the corresponding maximal Lyapunov exponent, then e^{β} is traditionally called the *joint spectral radius* (JSR).

Definition (Gian-Carlo Rota, W. Gilbert Strang, 1960)

The *joint spectral radius* (JSR) of a tuple of matrices $\mathcal{A} = (A_1, \ldots, A_m)$ is

$$\mathrm{JSR}(\mathcal{A}) := \lim_{n\to\infty} \max_{i_1,\ldots,i_n} \|A_{i_n} \cdots A_{i_1}\|^{\frac{1}{n}}.$$

Typical periodic maximization?

Conjecture (Mohsen Maesumi, 2008)

For Lebesgue-almost every m-tuple of $d \times d$ matrices, the JSR is attained by a shift-invariant measure periodic orbit for the shift.

It would be natural to expect uniqueness... However, this is false:

Theorem (J.B., Piotr Laskawiec, 2024)

There exists a nonempty (but tiny) open set $\mathcal{U}$ of pairs of 2×2 matrices such that if $(A_1, A_2) \in \mathcal{U}$, then the JSR is attained along two different periodic orbits, namely the orbits of the points

$$(121122)^\infty \quad \textit{and} \quad (221121)^\infty .$$

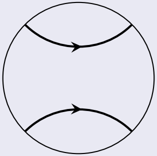
Pairs of 2×2 matrices

Theorem (Piotr Laskawiec, 2025)

There is a "big chunk" of the space of pairs of 2×2 matrices admitting a maximizing measure supported on a "simple" maximizing measure.

Theorem (J.B., Çagrı Sert, 2026?)

Maesumi conjecture holds for pairs of $\mathrm{SL}(2, \mathbb{R})$ matrices in "coparallel configuration":



Furthermore, maximizing measures are always unique and Sturmian.

Laskawiec Sert

Isometries and ergodic optimization

(H, d) = a noncompact metric space, $o \in H$ an arbitrary basepoint.

Definition

The *escape speed* of an isometry $f : H \to H$ is

$$\mathrm{ES}(f) := \lim_{n\to\infty} \frac{d(f^n(o), o)}{n}.$$

Definition

The *joint escape speed* of $\mathcal{F} = (f_1, \ldots, f_m)$ = a tuple of isometries of (H, d) is

$$\mathrm{JES}(\mathcal{F}) := \lim_{n\to\infty} \frac{1}{n} \max_{i_1,\ldots,i_n} d(f_{i_n} \circ \cdots \circ f_{i_1}(o), o).$$

= the maximal value of a subaddditive ergodic optimization problem associated to a one-step cocycle of isometries...

Optimizing Euclidean isometries

The group of isometries of the hyperbolic plane is isomorphic to the group

$$\mathrm{SL}^{\pm}(2,\mathbb{R}) = \{A \in \mathrm{Mat}(2 \times 2)\ ;\ \det A = \pm 1\}\,.$$

If Maesumi's conjecture is true for this group, then for typical tuples $\mathcal{F} = (f_1, \ldots, f_m)$ (with respect to Haar^m), the optimal ways of escaping to infinity should consists of periodic sequences.

Remark

By Furstenberg's theorem, *random* products typically escape to infinity with positive (but suboptimal) speed.

Optimizing Euclidean isometries

Assume that our isometries act $\mathcal{F} = (f_1, \ldots, f_m)$ act on Euclidean space $\mathbb{R}^d$.

Remark

Under mild (and typical) conditions on $\mathcal{F}$, *random* orbits $f_{j_1} \circ \cdots \circ f_{j_n}(o)$ satisfy a central limit theorem and follow a Brownian motion (in an appropriate limit): Tutubalin (1967) etc. In particular, the escape speed of Bernoulli measure is **zero**.

Proposition (Emmanuel Breuillard, Koji Fujiwara, 2021)

The joint escape speed $\mathrm{JES}(\mathcal{F})$ *is positive if and only if the* f_j*'s have no common fixed point.*

Breakdown of typically periodic optimization

Consider the simplest case of a pair of orientation-preserving isometries f_1, f_2 of $\mathbb{R}^2 = \mathbb{C}$. Impose the following (typical) hypotheses:

- f_j is not a translation:

$$f_j(z) = e^{i\alpha_j}(z - c_j) + c_j\,, \quad \alpha_j \notin 2\pi\mathbb{Z}\,.$$

- No common fixed points: $c_1 \neq c_2$, thus ensuring $\mathrm{JES}(f_1, f_2) > 0$.
- α_1, α_2, 2π are **rationally independent** (i.e. LI over $\mathbb{Q}$).

Then **no maximizing measure can be supported on a periodic orbit!**

$$f_{j_1} \circ \cdots \circ f_{j_n}(z) = e^{i(k\alpha_1 + \ell\alpha_2)} z + b$$

Main theorem

Theorem (J.B., Pablo Lessa, 2025?)

For a positive measure set of parameters (α_1, α_2), the maximizing measure is unique and Sturmian. This set is necessarily nowhere dense.

The proof involves reducing a **commutative** ergodic optimization problem over a (specific) **partially hyperbolic skew-product**.

First reduction: horizontal escape

Instead of maximizing $\sqrt{x^2+y^2}$, we can maximize x instead.

Lemma

$$\mathrm{JES}(f_1, f_2) = \lim_{n\to\infty} \frac{1}{n} \max_{j_1,\ldots,j_n} \left\langle f_{j_n} \circ \cdots \circ f_{j_1}(0), (1,0) \right\rangle.$$

Second reduction: PH skew-product

$$f_1(z) = e^{i\alpha_1} z + b_1\,, \qquad f_2(z) = e^{i\alpha_2} z + b_2\,.$$

$$\Omega := \{1,2\}^{\mathbb{N}}, \qquad \mathbb{T} := \frac{\mathbb{R}}{2\pi\mathbb{Z}}\,.$$

$$T\colon \Omega \times \mathbb{T} \to \Omega \times \mathbb{T}\,, \qquad T(\omega,\theta) := (\sigma(\omega), \theta + \alpha_{\omega_0})\,.$$

$$F\colon \Omega \times \mathbb{T} \to \mathbb{R}\,, \qquad F(\omega,\theta) := \mathrm{Re}(e^{i\theta} b_{\omega_0})\,.$$

Lemma

$$\mathrm{JES}(f_1, f_2) = \beta(F \mid T) := \sup_{\nu \in \mathcal{M}_T} \int_{\Omega\times\mathbb{T}} F \, d\nu\,.$$

A simplification: WLOG, $b_1 = b_2 = 1$, so

$$F(\omega,\theta) = \cos\theta\,.$$

Summary so far

We're now reduced to the following modified problem:

Consider the IFS acting on the unit circle S^1:

$$R_1(z) = e^{i\alpha_1} z\,, \quad R_2(z) = e^{i\alpha_2} z \qquad (\alpha_1, \alpha_2, 2\pi \text{ rationally indep.})$$

Given an initial point $z_0 \in S^1$ (say, $z_0 = 1$), we want to choose $\omega_0, \omega_1, \ldots \in \{1, 2\}^{\mathbb{N}}$ so to maximize the **average** value of

$$\mathrm{Re}\big(R_{\omega_{n-1}} \circ \cdots \circ R_{\omega_0}(z_0)\big)$$

What is the optimal strategy?

Instant gratification

Greedy strategy: Given $z_n \in S^1$, choose $\omega_n \in \{1, 2\}$ that maximizes $\mathrm{Re}(R_{\omega_n}(z_n))$ is maximal.

Instant gratification

Suppose $(\alpha_1, \alpha_2) \in [-\pi, \pi]^2$ are such that

$$\alpha_1\alpha_2 < 0\,, \quad |\alpha_1| + |\alpha_2| < \pi \quad (\text{“bowtie” region}).$$

Then orbits (z_n) of the greedy strategy remains in the arc

$$I := \left[-\tfrac{|\alpha_1|+|\alpha_2|}{2}, \tfrac{|\alpha_1|+|\alpha_2|}{2}\right] \subset (-\pi, \pi] \simeq \mathbf{T} \simeq S^1\,.$$

The greedy strategy is Sturmian

Recall $T(\omega, \theta) := (\sigma(\omega), \theta + \alpha_{\omega_0})$.

Under the conditions above ("bowtie"), there exists a T-invariant measure ν supported on the strip $\Omega \times I$, of the form $\nu = \mu_\gamma \times \mathrm{Leb}_I$, where μ_γ is the Sturmian measure on $\Omega = \{1, 2\}^{\mathbb{N}}$ with a proportion of 1's equal to $\gamma := \dfrac{|\alpha_2|}{|\alpha_1| + |\alpha_2|}$.

[DRAW FIGURE]

Main theorem, again

Theorem (B., Lessa)

Let $G \subsetneq \text{Bowtie} \subset [-\pi, \pi]^2$ *be the subset of parameters* (α_1, α_2) *for which the greedy measure is uniquely maximizing for the escape speed. Then:*

- *G has positive Lebesgue measure.*
- *G is nowhere dense (i.e.,* $\text{int}(\overline{G}) = \emptyset$*.)*

When greed is good

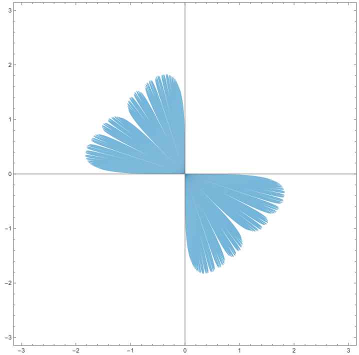
Theorem

Let (α_1, α_2) in the bowtie region be such that

$$\gamma := \frac{|\alpha_2|}{|\alpha_1| + |\alpha_2|} \notin \mathbb{Q},$$

$$|\alpha_1| + |\alpha_2| < C \left(\sum_{n=1}^{\infty} \frac{1}{n^2 |\sin(n\pi\gamma)|} \right)^{-1}$$

(where $C > 0$ is an explicit constant). Then $(\alpha_1, \alpha_2) \in G$ (greedy is uniquely maximal). In particular, $\mathrm{Leb}(G) > 0$.



The RHS is strictly positive for Lebesgue a.e. $\gamma \in [0, 1]$ (being Diophantine with exponent $\tau < 1$ suffices).

Delaying gratification is sometimes a good idea

If γ is super Liouville (very well approximable by rationals), then the greedy strategy is suboptimal.

[GO TO BOARD]

What do we gain with delaying gratification?

Lemma

If $\gamma \in [0,1]$ is "sufficiently Diophantine", then for every piecewise C^2 function $\phi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, the cohomological equation

$$\phi - c = \psi \circ R_\gamma - \psi$$

(where $c := \int \phi \, d\mathrm{Leb}$) admits a C^0 solution $\psi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$.

Proof.

Exercise on Fourier series. □

We can estimate $2\|\psi\|_{C^0}$, which is an upper bound for the relative payoff of "delayed gratification".

Directions for future research

1. Develop a theory of ergodic optimization beyond uniform hyperbolicity – applying at least to the partially hyperbolic skew products above.
2. Provide a more complete picture of the optimization of escape speed for composition of isometries.

Parabéns Alexander pelo seu excelente trabalho!