LYAPUNOV EXPONENTS FOR Mob($\mathbb{D}$)-COCYCLES: A PROOF OF OSELEDETS THEOREM IN DIMENSION 2

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Oseledets multiplicative ergodic theorem

Let $T : M \to M$ be an invertible measurable transformation, and μ be a T-invariant probability on M. Let $\pi : \mathcal{E} \to M$ be a finite-dimensional real vector bundle over M, endowed with a measurable Riemannian metric $\|\cdot\|$. Let $F : \mathcal{E} \to \mathcal{E}$ be a measurable vector bundle automorphism over T: this means that $\pi \circ F = T \circ \pi$ and the action $F_x : \mathcal{E}_x \to \mathcal{E}_{T(x)}$ of F on each fiber $\mathcal{E}_x = \pi^{-1}(x)$ is a linear isomorphism. We also call F a *cocycle* over T.

Example. (dynamical cocycle) Let $T : M \to M$ be a C^1 diffeomorphism on a Riemannian manifold M, $\mathcal{E}$ be the tangent bundle of M, and $F = DT$ be the derivative of T.

Example. (trivial bundle) Suppose $\mathcal{E} = M \times \mathbb{R}^d$. Then each F_x is a $d \times d$ matrix, that is, the cocycle corresponds to a map from M to the linear group $\mathbf{GL}(d, \mathbb{R})$.

Oseledets ergodic theorem states that, under an integrability condition, almost every fiber splits as a direct sum of subspaces such that the iterates $F_x^n : \mathcal{E}_x \to \mathcal{E}_{T^n(x)}$ have well-defined rates of exponential growth, in norm, restricted to each subspace:

Theorem 1 (Oseledets [Ose68]). *Assume* $\log^+ \|F_x\|$ *and* $\log^+ \|F_x^{-1}\|$ *are* μ*-integrable (where* $\log^+ = \max\{0, \log\}$*). Then for* μ*-almost every* $x \in M$*, there exist* $k(x) \in \mathbb{N}$*, real numbers* $\lambda_1(x) > \cdots > \lambda_{k(x)}$*, and a splitting* $\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$ *such that for every* $1 \le i \le k(x)$ *we have* $F_x(E_x^i) = E_{T(x)}^i$ *and*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|F_x^n(v_i)\| = \lambda_i(x) \quad \text{for all non-zero } v_i \in E_x^i\,.$$

Moreover, the Lyapunov exponents $\lambda_i(x)$*, and the Oseledets subspaces* E_x^i *are unique* μ*-almost everywhere, and they depend measurably on* x*.*

We are going to prove Oseledets theorem in the case when the vector bundle is 2-dimensional. For simplicity we also suppose that it is a trivial bundle, that is, $\mathcal{E} = M \times \mathbb{R}^2$. Then each F_x is given by a 2×2 matrix. We take these matrices to be in $\mathbf{SL}(2, \mathbb{R})$, that is, to have determinant 1. This is not a restriction because the validity of the theorem is not affected if we multiply F by some non-zero function θ (as long as the integrability condition is preserved) : the Oseledets subspaces remain the same, and one adds the Birkhoff average of $\log|\theta|$ to the Lyapunov exponents.

There are several proofs of Oseledets theorem in the literature, besides the original one. See for instance [Mañ87, Chapter 4]. Another proof of the 2-dimensional case can be found in [You95].

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Angles between Oseledets directions. As a complement to theorem 1, we also prove that the angles between the spaces of the Oseledets splitting $\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$ are sub-exponential, in the sense made precise by the following:

Proposition 2. *Let $\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^{k(x)}$ be the splitting given by Oseledets theorem. Fix an integer k with $2 \le k \le \dim \mathcal{E}$ and let $J_1 \sqcup J_2$ be a partition of the set of indices $\{1, \dots, k\}$ into two disjoint non-empty sets. If $k(x) = k$, denote $\widehat{E}_x^i = \oplus_{j \in J_i} E_x^j$ for $i = 1$, 2. Then, for μ-a.e. x satisfying $k(x) = k$, we have*

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \sin \sphericalangle\big(\widehat{E}^1_{T^n(x)}, \widehat{E}^2_{T^n(x)}\big) = 0.$$

We will assume Oseledets theorem and deduce proposition 2 from two simple results, proposition 3, from ergodic theory, and proposition 4, from linear algebra.

Proposition 3. *Let $B \subset M$ be a T-invariant positive measure set and $\varphi : B \to \mathbb{R}$ be a measurable function such that $\varphi \circ T - \varphi$ is integrable* [1]. *Then $\lim_{n\to\infty} \varphi(T^n(x))/n = 0$ for a.e. $x \in B$.*

Proof. Let $\psi = \varphi \circ T - \varphi$ and let $\hat\psi$ be the limit of the Birkhoff averages of ψ. Then $(\varphi \circ T^n)/n \to \hat\psi$ a.e. in B. In particular, $\hat\psi(x) \neq 0$ implies $|\varphi(T^n(x))| \to \infty$. But, by the Poincaré's recurrence theorem, the set of points $x \in B$ which satisfy the latter condition has zero measure. Therefore $\hat\psi = 0$ a.e. in B. ☐

Proposition 4. *Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a invertible linear tranformation and let v, w be non-zero vectors. Then*

$$\frac{1}{\|L\|\,\|L^{-1}\|} \le \frac{\sin \sphericalangle(Lv, Lw)}{\sin \sphericalangle(v, w)} \le \|L\|\,\|L^{-1}\|.$$

Proof. Recall that for any $\alpha \in \mathbb{R}$, $\|w + \alpha v\| \ge \|w\| \sin \sphericalangle(v, w)$, with equality when $\alpha = \langle v, w\rangle / \|v\|^2$.

Let $\beta = \langle Lv, Lw\rangle / \|Lv\|^2$ and $z = w + \beta v$. Then $\|z\| \ge \|w\| \sin \sphericalangle(v, w)$ and, on the other hand, $\|Lz\| = \|Lw\| \sin \sphericalangle(Lv, Lw)$. Therefore

$$\sin \sphericalangle(Lv, Lw) = \frac{\|Lz\|}{\|Lw\|} \ge \frac{\|L^{-1}\|^{-1} \cdot \|z\|}{\|L\| \cdot \|w\|} \ge \frac{\sin \sphericalangle(v, w)}{\|L\|\,\|L^{-1}\|},$$

proving the first inequality. The second one is analogous. ☐

Proof of proposition 2. Let $B \subset M$ be the T-invariant set of points x where $k(x) = k$. The function $\varphi : B \to \mathbb{R}$ defined by $\varphi(x) = \log \sin \sphericalangle(\widehat{E}_x^1, \widehat{E}_x^2)$ is measurable. By proposition 4, we have $|\varphi(Tx) - \varphi(x)| \le \log \|F_x\| + \log \|F_x^{-1}\|$, which, by the hypothesis of the Oseledets theorem, is integrable. So proposition 3 gives $\varphi(T^n(x))/n \to 0$ for a.e. $x \in B$. ☐

Automorphisms of the disk

$\mathrm{Mob}(\mathbb{D})$ is the set of all *automorphisms* of the unit disk $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$, that is, all conformal diffeomorphisms $f : \mathbb{D} \to \mathbb{D}$. The canonical form of an automorphism $f \in \mathrm{Mob}(\mathbb{D})$ is

$$f(z) = \beta \varphi_\alpha(z), \quad \text{where} \quad \varphi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}, \ \alpha \in \mathbb{D},\ |\beta| = 1.$$

[1] With the stronger assumption "φ is integrable" the proposition would be a direct consequence of Birkhoff's theorem.

The *hyperbolic metric* on the disk is given by

$$d\rho = \frac{2\,|dz|}{1-|z|^2}\,.$$

Since the straight lines through the origin are geodesics, we have

$$\rho(z,0) = 2\int_0^{|z|} \frac{dr}{1-r^2} = \log\frac{1+|z|}{1-|z|} = 2\,\text{arctgh}\,|z|\,. \tag{1}$$

Using the fact that the hyperbolic metric is invariant under every automorphism, we may deduce its general expression

$$\rho(z_1,z_2) = \rho(\varphi_{z_2}(z_1),\varphi_{z_2}(z_2)) = \rho\big(\frac{z_1-z_2}{1-z_1\overline{z}_2}\,,0\big) = 2\,\text{arctgh}\,\frac{|z_1-z_2|}{|1-z_1\overline{z}_2|}\,. \tag{2}$$

The proof of the following formula may be found in [Nic89, page 12].

Proposition 5. *Let* $f \in \text{Mob}(\mathbb{D})$ *and* $\xi \in \partial\mathbb{D}$. *Then*

$$|f'(\xi)| = \frac{1-|f(0)|^2}{|\xi - f^{-1}(0)|^2}\,.$$

We denote by $\text{Mob}(\mathbb{H})$ the set of automorphisms of the complex half-plane $\mathbb{H} = \{z \in \mathbb{C};\ \text{Im}\,z > 0\}$. Each $f \in \text{Mob}(\mathbb{H})$ has the form

$$f(z) = \frac{az+b}{cz+d}\,,\quad a,b,c,d \in \mathbb{R},\ ad-bc \neq 0.$$

We also fix a conformal equivalence $h : \mathbb{H} \to \mathbb{D}$, for instance,

$$h(z) = \frac{i-z}{i+z}\,. \tag{3}$$

Automorphisms *versus* matrices

There is a natural group isomorphism between $\text{Mob}(\mathbb{H})$ and the projective linear group $\mathbf{PSL}(2,\mathbb{R}) = \mathbf{PGL}(2,\mathbb{R})$, as follows. If

$$f(z) = \frac{az+b}{cz+d} \in \text{Mob}(\mathbb{H}) \quad \text{and} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R})$$

then $f|_{\partial\mathbb{H}} = f|_{\mathbb{R}\cup\{\infty\}}$ describes the action of A on the projective space $\mathbb{R}\mathbf{P}^1$: the matrix A maps the line with co-slope x (i.e. the line containing the vector $(x,1)$) to the line with co-slope $(ax+b)/(cx+d)$.

Remark. Identifying $\partial\mathbb{H}$ with $\mathbb{R}\mathbf{P}^1$ via co-slopes in this way, we see that the map $h : \mathbb{H} \to \mathbb{D}$ induces a homeomorphism $\phi : \mathbb{R}\mathbf{P}^1 \to \partial\mathbb{D}$ defined by $[(\cos\theta,\sin\theta)] \mapsto e^{-2i\theta}$.

Using the conformal equivalence $h : \mathbb{H} \to \mathbb{D}$ we find a corresponding isomorphism between $\text{Mob}(\mathbb{D})$ and $\mathbf{PSL}(2,\mathbb{R})$. Denote by $f_A \in \text{Mob}(\mathbb{D})$ the automorphism associated to the projective class $[A] = \{\pm A\}$ of each $A \in \mathbf{SL}(2,\mathbb{R})$. We leave it to the reader to check that $f_A(0) = 0$ if and only if A is a rotation.

Remark. Corresponding to the canonical form $f(z) = \beta\varphi_\alpha(z)$ for the elements of $\text{Mob}(\mathbb{D})$ we have a canonical form of $A = \pm R_1 H R_2$, where R_1 and R_2 are rotation matrices and

$$H = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}.$$

Proposition 6. *Given $A \in \mathbf{SL}(2,\mathbb{R})$ and $v \in \mathbb{R}^2$ with $|v| = 1$, let ξ be the point of $\partial\mathbb{D}$ associated to the direction of v under the homeomorphism ϕ. Then*

$$|Av| = |f'_A(\xi)|^{-1/2}.$$

A heuristic argument using infinitesimals goes as follows. A maps the unit circle onto an ellipse, preserving area. The slice of the unit disk corresponding to the element of angle $d\theta$ around v, whose area is $2d\theta$, is mapped to a slice of the ellipse with area $2\,|Av|^2\,d\varphi$. Therefore, $|f'_A(\xi)| = d\varphi/d\theta = |Av|^{-2}$. A formal proof follows.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $g(z) = \dfrac{az+b}{cz+d}$ be the corresponding element of $\mathrm{Mob}(\mathbb{H})$. Then $f_A = h \circ g \circ h^{-1}$. Consider a vector $w = (x,1) \in \mathbb{R}^2$ and let $\xi \in \partial\mathbb{D}$ be associated to the direction of w. We may consider the map h in (3) as an automorphism of the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{C}\mathbf{P}^1 = \mathbf{S}^2$. Then h is an isometry relative to the metric defined on the sphere by

$$\|dz\|_z = \frac{2\,|dz|}{|z|^2+1}.$$

So, considering f_A and g also as maps on the sphere,

$$\|D(f_A)_\xi\| = \left\|Dh_{g(x)} \circ Dg_x \circ D(h^{-1})_\xi\right\| = \|Dg_x\|.$$

Now, $\|D(f_A)_\xi\| = |f'_A(\xi)|$ and

$$\begin{aligned} \|Dg_x\| &= |g'(x)|\,\frac{\|\cdot\|_{g(x)}}{\|\cdot\|_x} = \frac{|ad-bc|}{|cx+d|^2}\,\frac{|x|^2+1}{|g(x)|^2+1} = \\ &= \frac{|x|^2+1}{|ax+b|^2+|cx+d|^2} = \frac{|w|^2}{|Aw|^2}. \end{aligned}$$

Normalizing the vector w we obtain the relation in the statement. □

Given $A \in \mathbf{SL}(2,\mathbb{R})$ which is not a rotation, let $s(A) \in \mathbb{R}\mathbf{P}^1$ the direction that is most contracted and $u(A) \in \mathbb{R}\mathbf{P}^1$ the direction the direction that is most expanded by A. For simplicity, we shall write also $u(A)$, $s(A)$ for $\phi(u(A))$, $\phi(s(A))$.

Corollary 7. *We have $s(A) = \dfrac{f_A^{-1}(0)}{\left|f_A^{-1}(0)\right|}$ and $u(A) = -s(A)$.*

Proof. Putting propositions 5 and 6 together we get

$$|Av| = |f'_A(\xi)|^{-1/2} = \frac{\left|\xi - f_A^{-1}(0)\right|}{\sqrt{1-|f_A(0)|^2}}.$$

Thus, the most contracted direction corresponds to the point $\xi \in \partial\mathbb{D}$ closest to $f_A^{-1}(0)$, which is $f_A^{-1}(0)/|f_A^{-1}(0)|$. Analogously for the most expanded direction. □

Corollary 7 also shows that the directions $s(A)$, $u(A) \in \mathbb{R}\mathbf{P}^1$ are orthogonal.

Proposition 8. $\|A\| = \sqrt{\frac{1+|f_A(0)|}{1-|f_A(0)|}} = \exp(\frac{1}{2}\rho(f_A(0),0))$ *for all $A \in \mathbf{SL}(2,\mathbb{R})$.*

Proof. If A is a rotation then $\|A\| = 1$ and $f_A(0) = 0$, and the equalities are clear. Otherwise, if A is not a rotation, proposition 6 gives

$$\|A\| = |A\,u(A)| = |f'_A(u(A))|^{-1/2} = \frac{\left|u(A) - f_A^{-1}(0)\right|}{\sqrt{1-|f_A(0)|^2}}.$$

Using proposition 5 and noting that $|f_A^{-1}(0)| = |f_A(0)|$, we conclude that

$$\|A\| = \frac{1 + |f_A^{-1}(0)|}{\sqrt{1 - |f_A(0)|^2}} = \frac{\sqrt{1 + |f_A(0)|}}{\sqrt{1 - |f_A(0)|}}$$

as claimed. □

Lyapunov exponents for Mob($\mathbb{D}$)**-Cocycles.** An $\mathbf{SL}(2,\mathbb{R})$-cocycle naturally induces a $\mathbf{PSL}(2,\mathbb{R})$-cocycle and, conversely, a $\mathbf{PSL}(2,\mathbb{R})$-cocycle can always be lifted to a $\mathbf{SL}(2,\mathbb{R})$-cocycle [2]. By definition, the Lyapunov exponents of a $\mathbf{PSL}(2,\mathbb{R})$-cocycle are those of any such lift, the choice being irrelevant. Using $\mathbf{PSL}(2,\mathbb{R}) = \mathrm{Mob}(\mathbb{D})$ we get a notion of Lyapunov exponent for Mob($\mathbb{D}$)-cocycles. This can be made explicit using proposition 8: writing $f_x = f_{F_x}$, and $f_x^n = f_{F_x^n} = f_{T^{n-1}x} \circ \cdots \circ f_x$ for each n, we find

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|F_x^n\| = \lim_{n\to\pm\infty} \frac{1}{2n} \rho(f_x^n(0), 0).$$

In other words, *the Lyapunov exponents correspond to the rates of growth of the hyperbolic distance from $f_x^n(0)$ to the origin.* So, in this context the theorem of Oseledets becomes:

Theorem 9. *Let $T : (X,\mu) \to (X,\mu)$ be an invertible ergodic transformation and $f : X \to \mathrm{Mob}(\mathbb{D})$ be a measurable map such that*

$$\int_X \rho(f_x(0), 0)\, d\mu(x) < \infty. \tag{4}$$

Then there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \rho(f_x^n(0), 0) = 2\lambda \quad \text{for } \mu\text{-almost every } x \in X. \tag{5}$$

Furthermore, if $\lambda > 0$ there are measurable maps $w^s, w^u : X \to \partial\mathbb{D}$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \log |(f_x^n)'(z)| = \begin{cases} 2\lambda & \text{if } z = w^s(x) \\ -2\lambda & \text{if } z \in \overline{\mathbb{D}} \text{ with } z \neq w^s(x), \end{cases}$$

$$\lim_{n\to-\infty} \frac{1}{n} \log |(f_x^n)'(z)| = \begin{cases} -2\lambda & \text{if } z = w^u(x) \\ 2\lambda & \text{if } z \in \overline{\mathbb{D}} \text{ with } z \neq w^u(x). \end{cases}$$

If $\lambda = 0$ then

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |(f_x^n)'(z)| = 0 \text{ for all } z \in \overline{\mathbb{D}}.$$

Remark. (1) In view of (1), the relation (5) is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \log(1 - |f_x^n(0)|) = 2\lambda \quad \text{for } \mu\text{-almost every } x.$$

(2) The contents of (4) and (5) do not change if we replace the origin by any other point in the open disk.
(3) If the system is not ergodic λ is a function of x but, otherwise, the theorem remains valid.

[2]This is one place where it is simpler to work on a trivial bundle. For general vector bundles, this lift can be done locally, and all that follows is easily adapted.

Proof of the Oseledets theorem in dimension 2

We shall use Kingman's sub-additive ergodic theorem in the following form:

Theorem 10 ([Kin68]). *Assume T is ergodic. If $(\varphi_n)_{n=1,2,\dots}$ is a sequence of functions such that φ_1^+ is integrable and $\varphi_{m+n} \le \varphi_m + \varphi_n \circ T^m$ for all m, $n \ge 1$. Then $\frac{1}{n}\varphi_n$ converges a.e. to some $c \in \mathbb{R} \cup \{-\infty\}$.*

Define $\varphi_n(x) = \rho(f_x^n(0), 0)$. Then $\varphi_{m+n} \le \varphi_m + \varphi_n \circ T^m$, by the triangle inequality. Using theorem 10 we get that $\frac{1}{n}\varphi_n$ converges μ-almost everywhere to a constant 2λ. Since $\varphi_n \ge 0$, $\lambda \ge 0$. This proves (5).

Define $w_n(x) = (f_x^n)^{-1}(0)$ for every integer n. Notice that, by the invariance of the hyperbolic metric, $\rho(w_n(x), 0) = \rho(f_x^n(0), 0)$. Using (1) we get, for a.e. x,

$$\lim_{n\to+\infty} \frac{1}{n} \log(1 - |w_n(x)|) = -2\lambda \tag{6}$$

Suppose that $\lambda > 0$. Then the distance from w_n to the origin goes to infinity, which means that w_n converges to the boundary of $\mathbb{D}$ as $n \to \infty$.

Lemma 11. *We have $\limsup\limits_{n\to+\infty} \frac{1}{n} \log|w_{n+1}(x) - w_n(x)| \le -2\lambda$ for μ-a.e. x.*

Proof. Since the hyperbolic metric is invariant under automorphisms,

$$\begin{aligned}\rho(w_{n+1}, w_n) &= \rho((f_x^n)^{-1} \circ (f_{T^n x})^{-1}(0), (f_x^n)^{-1}(0)) = \\ &= \rho((f_{T^n x})^{-1}(0), 0) = \rho(f_{T^n x}(0), 0).\end{aligned} \tag{7}$$

The idea of the proof is that if $\rho(f_{T^n x}(0), 0)$ is not too big, that is, if w_{n+1} and w_n are not too far away from each other in terms of the hyperbolic metric, then the Euclidian distance between w_{n+1} and w_n will have to be exponentially small, since $w_n \to \partial\mathbb{D}$ exponentially fast. Write $b_n(x) = f_{T^n(x)}(0)$, for simplicity. For a.e. x, we have

$$\frac{1}{n}\rho(b_n(x), 0) \to 0\,. \tag{8}$$

This follows from proposition 3 applied to the function $\varphi(x) = \rho(f_x(0), 0)$, which, by assumption (4), is integrable. Fix x in the full measure set where (6) and (8) hold. In view of (1)–(2) the equality (7) implies

$$\frac{|w_{n+1} - w_n|}{|1 - w_n \overline{w}_{n+1}|} = |b_n|$$

or, equivalently,

$$\begin{aligned}|w_{n+1} - w_n| = |b_n|\,|1 - w_n\overline{w}_{n+1}| &= |b_n| \left|1 - |w_n|^2 + w_n(\overline{w}_n - \overline{w}_{n+1})\right| \\ &\le |b_n| \left(1 - |w_n|^2 + |w_n|\,|w_{n+1} - w_n|\right)\,.\end{aligned}$$

That is,

$$|w_{n+1} - w_n| \le \frac{|b_n|(1 - |w_n|^2)}{1 - |b_n|\,|w_n|}\,.$$

Since $|w_n| < 1$ and $|b_n| < 1$, the last inequality implies

$$|w_{n+1} - w_n| < \frac{1 - |w_n|^2}{1 - |b_n|} < \frac{2\,(1 - |w_n|)}{1 - |b_n|}.$$

The condition (8) is equivalent to $\frac{1}{n}\log(1-|b_n|)\to 0$. Combining this with (6) and the inequality above, we conclude

$$\limsup_{n\to+\infty}\frac{1}{n}\log|w_{n+1}-w_n|\le -2\lambda.$$

□

The lemma implies that $w_n(x)=(f_x^n)^{-1}(0)$ is a Cauchy sequence for almost every x. Let $w^s(x)\in\partial\mathbb{D}$ be the limit. Notice that, by corollary 7,

$$w^s(x)=\lim_{n\to\infty}w_n(x)=\lim_{n\to\infty}\frac{w_n(x)}{|w_n(x)|}=\lim_{n\to\infty}s(f_x^n).$$

Let us show how to compute the growth rate of $\log|(f_x^n)'(z)|$ for $z\in\partial\mathbb{D}$. By proposition 5, and using $|f_x^n(0)|=|(f_x^n)^{-1}(0)|$, we have

$$|(f_x^n)'(z)|=\frac{1-|f_x^n(0)|^2}{|z-(f_x^n)^{-1}(0)|^2}=(1+|w_n|)\frac{1-|w_n|}{|z-w_n|^2}.$$

Using (6) we deduce

$$\lim_{n\to+\infty}\frac{1}{n}\log|(f_x^n)'(z)|=-2\lambda-2\lim_{n\to+\infty}\frac{1}{n}\log|z-w_n|\,. \tag{9}$$

For all $z\neq w^s(x)$ this gives that the limit is -2λ.

Now consider the case $z=w^s(x)$. Since $|w_n-w^s|\ge 1-|w_n|$, we have $\liminf\frac{1}{n}\log|w^s-w_n|\ge -2\lambda$. On the other hand, take $0<\varepsilon<2\lambda$. By lemma 11, we have $|w_{j+1}-w_j|\le e^{(-2\lambda+\varepsilon)j}$ if j is large enough. Hence

$$|w^s-w_n|\le\sum_{j=n}^{\infty}|w_{j+1}-w_j|\le\frac{e^{(-2\lambda+\varepsilon)j}}{1-e^{-2\lambda+\varepsilon}}\,,$$

and so $\limsup\frac{1}{n}\log|w^s-w_n|\le -2\lambda+\varepsilon$. This proves that $\lim\frac{1}{n}\log|w^s-w_n|=-2\lambda$. Substituting in (9), we get $\lim\frac{1}{n}\log|(f_x^n)'(z)|=2\lambda$.

Now we do the corresponding calculation for $z\in\mathbb{D}$. By the invariance of the hyperbolic metric under f_x^n, we have

$$|(f_x^n)'(z)|=\frac{1-|f_x^n(z)|^2}{1-|z|^2}=\frac{1-|w_n|^2}{1-|z|^2}\,. \tag{10}$$

It follows, using (6), that $\lim\frac{1}{n}\log|(f_x^n)'(z)|=2\lambda$.

The statements about w^u follow by symmetry, considering the inverse cocycle.

At last, we consider the case $\lambda=0$. If $z\in\partial\mathbb{D}$, then using (9) and $1-|w_n|\le|z-w_n|\le 2$, we get $\lim\frac{1}{n}\log|(f_x^n)'(z)|=0$. If $z\in\mathbb{D}$ then we simply use (10).

The proof of theorem 9 is complete.

Remark. Using proposition 3, it is easy to prove that if $\lambda>0$ then

$$\lim_{n\to\pm\infty}\frac{1}{n}\log|w^u(T^nx)-w^s(T^nx)|=0.$$

This corresponds to the fact that the angles between Oseledets directions are sub-exponential.

Higher dimensions. The Oseledets theorem for the disk $\mathbb{D}$ extends directly to the hyperbolic ball $\mathbb{B}^n$ of dimension $n \geq 2$. The formula (2) can be generalized as follows. Given two points in $\mathbb{B}^n$ consider the disk that contains both and the origin. Since this disk is isometric to $\mathbb{D}$, we may use (2) to compute the distance between the points. In the case $n = 3$ there is an isomorphism between the groups $\mathrm{Mob}(\mathbb{B}^3)$ and $\mathbf{PSL}(2, \mathbb{C})$, induced by identifying $\partial \mathbb{B}^3 = \mathbb{C}\mathbf{P}^1$. It is an extension of the isomorphism between $\mathrm{Mob}(\mathbb{D})$ and $\mathbf{PSL}(2, \mathbb{R})$ that we used above, and all the relations between real matrices and automorphisms of the disk that we proved extend to complex matrices and automorphisms of the ball. We ignore whether there is a nice matrix representation of the groups $\mathrm{Mob}(\mathbb{B}^n)$ for $n \geq 4$.

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