Optimization of Lyapunov Exponents of Matrix Cocycles

Joint work with Michał Rams (Warsaw)

Ergodic Optimization and Related Fields – IME-USP

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BASIC SETTING

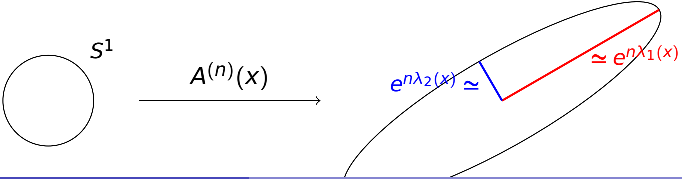
Lyapunov exponents (2×2 case)

Let $T\colon X \to X$ be a homeomorphism of a compact space. Let $A\colon X \to \mathrm{GL}(2,\mathbb{R})$ be continuous. The pair (T, A) is called a *cocycle*. Consider "Birkhoff-like" products:

$$A^{(n)}(x) := A(T^{n-1}x) \cdots A(Tx)A(x).$$

The *upper and lower Lyapunov exponents* at the point x (if they exist) are:

$$\lambda_1(x) := \lim_{n\to+\infty} \frac{1}{n} \log \left\| A^{(n)}(x) \right\|$$
$$\lambda_2(x) := \lim_{n\to+\infty} \frac{1}{n} \log \left\| [A^{(n)}(x)]^{-1} \right\|^{-1}$$



Oseledets Theorem

Theorem (Invertible 2×2 Oseledets)

Let μ be an ergodic probability measure for T. Then there exist $\lambda_1(\mu) \geq \lambda_2(\mu)$ such that $\lambda_i(x) = \lambda_i(\mu)$ for μ-a.e. x.

Moreover, if $\lambda_1(\mu) > \lambda_2(\mu)$ then there exists an "Oseledets splitting" $\mathbb{R}^2 = E_1(x) \oplus E_2(x)$, defined for μ-a.e. x and such that for each $i = 1, 2$:

- *the spaces E_i vary measurably w.r.t. x;*
- *the spaces E_i are equivariant: $A(x)(E_i(x)) = E_i(Tx)$;*
- $\displaystyle\lim_{n \to \pm\infty} \frac{1}{n} \log \|A^{(n)}(x) v\| = \lambda_i(\mu), \quad \forall v \in E_i(x) \setminus \{0\}$ *($i = 1, 2$).*

☞ We'll see much more on Anthony Quas lectures.

Optimization of Lyapunov exponents

Fixed cocycle (T, A) as above, let:

$$\lambda_1^{\top} := \sup_{\mu \in \mathcal{M}_{\text{erg}}(T)} \lambda_1(\mu), \qquad \lambda_1^{\perp} := \inf_{\mu \in \mathcal{M}_{\text{erg}}(T)} \lambda_1(\mu).$$

(Rem.: The λ_2 case can be reduced to the λ_1 case by inverting time and sign.)

Questions:

- Are the sup and inf above attained? That is, are there $\mu \in \mathcal{M}_{\text{erg}}(T)$ such that $\lambda_1(\mu) = \lambda_1^{\top}$ or $\lambda_1^{\perp}$?
- How are those measures? What are their entropies, dimensions, etc?
- Are they unique?
- Are they characterized by their support?

Attempt to reduce to the traditional theory

Note that λ_1 is the limit of the Birkhoff averages of the function

$$f(x) := \log \left\| A(x)|_{E_1(x)} \right\|$$

where E_1 is the first Oseledets space. Can we apply the traditional ergodic optimization theory to f?

The first problem is that in general f is not defined everywhere and it's only measurable.

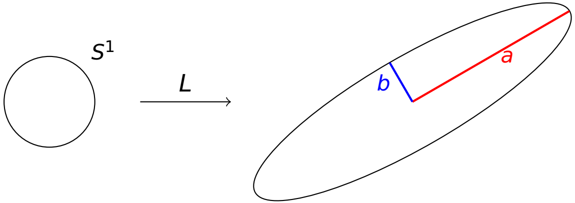
But there are situations where f is more regular...

Dominated cocycles

The cocycle $(T, : X \to X, A \colon X \to \mathrm{GL}(2,\mathbb{R}))$ is called *dominated* if there exist $c > 0$ and $\delta > 0$ such that

$$\mathrm{nc}(A^{(n)}(x)) > ce^{\delta n} \quad \forall\, x \in X,\ n \geq 0\,,$$

where $\mathrm{nc}(L) := \|L\|\,\|L^{-1}\|$ is the *non-conformality* of the matrix L.



$\boxed{\mathrm{nc}(L) = a/b}$

Example (Positive matrices)

If all matrices $A(x)$ have strictly positive entries then the cocycle is dominated.

Dominated cocycles

If the cocycle (T, A) is dominated then there exist a **continuous** splitting $\mathbb{R}^2 = E_1(x) \oplus E_2(x)$, defined for **every** $x \in X$ that extends the Oseledets splitting.

In particular, for dominated cocycles the function $f(x) := \log \left\| A(x)|_{E_1(x)} \right\|$ is **continuous** (and not only measurable).

Remark about terminology: It can be shown that E_1 "*dominates*" E_2, i.e., $\exists\ c > 0$ and $\delta > 0$ such that

$$\frac{\|A^{(n)}(x)|_{E_1(x)}\|}{\|A^{(n)}(x)|_{E_2(x)}\|} > ce^{\delta n} \quad \forall\, x \in X,\ n \geq 0\,,$$

So $E_1 \oplus E_2$ is a "*dominated splitting*".

Conversely, existence of such a dominated splitting (easily) implies that the cocycle is dominated as we have defined.

A SPECIAL SITUATION:

ONE-STEP COCYCLES

One-step cocycles

A cocycle (T, A) is *one-step* if:

- $T \colon X \to X$ is the two-sided full shift on $k \geq 2$ symbols, so
$$X = \Big\{ (\xi_i)_{i \in \mathbb{Z}};\ \xi_i \in \{1, 2, \dots, k\} \Big\}.$$

- $A \colon X \to \mathrm{GL}(2, \mathbb{R})$ only depends on the zeroth symbol. In other words, there is a list of matrices $A_1, \dots, A_k$ such that
$$\xi = (\xi_i)_{i \in \mathbb{Z}} \ \Rightarrow \ A(\xi) = A_{\xi_0}$$

  (In particular, A is locally constant.)

Equivalent setting: *Linear IFS's (iterated function systems)*.

Joint spectral radius and subradius

Equivalent *elementary* definitions of $\lambda_1^\top$, $\lambda_1^\perp$ **for one-step cocycles**:

$$\lambda_1^\top = \lim_{n\to\infty} \frac{1}{n} \log \sup_{i_1,\dots,i_n} \|A_{i_1} \dots A_{i_n}\| \,,$$
$$\lambda_1^\perp = \lim_{n\to\infty} \frac{1}{n} \log \inf_{i_1,\dots,i_n} \|A_{i_1} \dots A_{i_n}\| \,.$$

For one-step cocycles, the numbers $e^{\lambda_1^\top}$ and $e^{\lambda_1^\perp}$ are known as *joint spectral radius* (Rota, Strang 1960) and *joint spectral subradius* (Gurvits, 1995) of the set of matrices $\{A_1, \dots, A_k\}$.

Applications in wavelets, control theory . . .

The finiteness conjecture on the Joint spectral radius

Finiteness conjecture (1995): *There always exists a Lyapunov-maximizing measure supported on a periodic orbit.*

The conjecture was shown to be **false** by Bousch and Mairesse (2002); their counterexamples are pairs of matrices in $\mathrm{GL}(2, \mathbb{R})$ such that the maximizing measures are **sturmian** but not periodic.

☞ We'll see much more about this on Mark Pollicott's talk, I guess.

Domination for one-step cocycles

We have already seen two equivalent definitions of domination for general (2×2) cocycles.

Let us see two extra characterizations of domination, this time specializing to **one-step** cocycles.

1. An one-step cocycle is dominated iff $\boxed{(\lambda_1 - \lambda_2)^\perp > 0}$, where

$$(\lambda_1 - \lambda_2)^\perp(A) := \inf_{\mu \in \mathcal{M}_{\mathrm{erg}}(T)} (\lambda_1(A, \mu) - \lambda_2(A, \mu)).$$

2. Geometrical characterization ...

Domination for one-step cocycles: geometrical viewpoint

Let $\mathbb{P}^1$ be the projective space (lines through the origin in $\mathbb{R}^2$).

A *multicone* for $\{A_1, \ldots, A_k\}$ is an open set $M \subsetneq \mathbb{P}^1$ such that

- M has a finite number of connected components, and they have disjoint closures.
- $\forall\ i$, the closure of the image $A_i(M)$ is contained in M.

Example

A_i positive $\Rightarrow$ the first quadrant is a multicone.

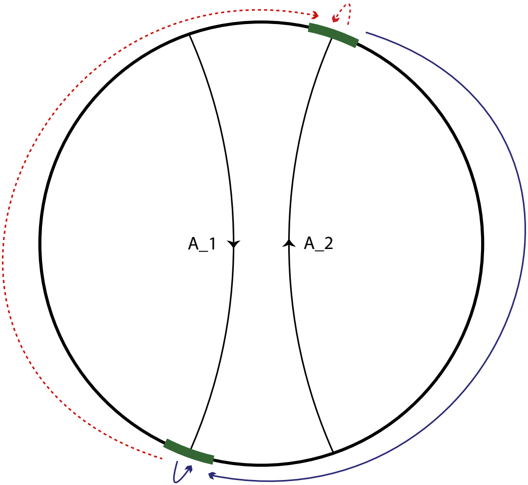
Theorem (Avila, B., Yoccoz)

An one-step cocycle is dominated if and only if it has a multicone.

Consequence: domination is an open condition.

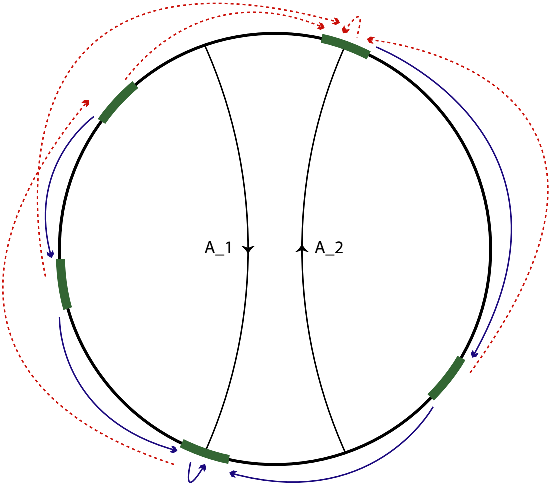
Example where the multicone is not a cone

The simplest example where the multicone is not connected (and cannot be replaced by a connected multicone) is:



A more complicated example

A (non-obvious) example where the multicone has at least 5 components:



Non-additive thermodynamic formalism

☞ If (T, A) is a dominated one-step cocycle then the sequence of functions

$$\phi_n(\xi) = \log \|A^{(n)}(\xi)\| = \log \|A_{\xi_{n-1}} \dots A_{\xi_1} A_{\xi_0}\|$$

is "almost-additive" and "regular" so the results of Godofredo Iommi's lecture apply: "zero temperature" limits are optimizing measures.

REVEALING FUNCTIONS

FOR DOMINATED ONE-STEP COCYCLES

Some of these terms should be familiar to many of you:

- Mañé Lemma
- revealing functions
- (calibrated) sub-actions
- Barabanov norms

Barabanov functions

Assume given an one-step dominated cocycle, with a multicone M. Let $\overrightarrow{M} := \{v \in \mathbb{R}^2;\ v = 0 \text{ or } \mathbb{R} \cdot v \in M\}$ be the "support" of M.

Proposition (Existence of a Barabanov function)

There exists a continuous function $|||\cdot||| : \overrightarrow{M} \to [0, \infty)$ *such that for every* $v \in \overrightarrow{M}$, $t \in \mathbb{R}$ *we have*

- $|||tv||| = |t|\ |||v|||$ *(homogeneity);*
- $\max_{i \in \{1,\dots,k\}} |||A_i v||| = e^{\lambda_1^\top} |||v|||$ *(main property).*

We call $|||\cdot|||$ an *upper Barabanov function*.

Consequence: For any $v \in \overrightarrow{M}$, we can find recursively symbols $i_1, i_2, \dots \in \{1, \dots, k\}$ such that

$$|||A_{i_n} \cdots A_{i_1}(v)||| = e^{n\lambda_1^\top} |||v|||.$$

Barabanov functions

Upper Barabanov function:

$$\|\!|tv|\!\| = |t|\ \|\!|v|\!\|, \qquad \max_{i \in \{1,\dots,k\}} \|\!|A_i v|\!\| = e^{\lambda_1^\top} \|\!|v|\!\|.$$

- Actually (even without domination) there is a true norm $\|\!|\cdot|\!\|$ in $\mathbb{R}^2$ with the properties above; it is called *Barabanov norm* (1988).
- There is also a *lower Barabanov function* $\|\!|\cdot|\!\|'$ such that

$$\|\!|tv|\!\| = |t|\ \|\!|v|\!\|, \qquad \min_{i \in \{1,\dots,k\}} \|\!|A_i v|\!\| = e^{\lambda_1^\perp} \|\!|v|\!\|.$$

In this case the domination hypothesis is really needed.

Construction of Barabanov functions

Our construction of Barabanov functions is similar to the traditional construction of revealing functions. The only “new” fact we use is that the Hilbert metric is contracted.

Proof of the Proposition:

- The desired function is (essentially) the fixed point of an operator acting on $C(M)$ mod constants.
- The operator preserves a compact subspace of functions (those satisfying a bound on the Lipschitz constant w.r.t. the Hilbert metric on M.).
- We apply Schauder fixed point theorem.

Mather sets

Proposition

If an one-step cocycle is ***dominated*** *then there are nonempty compact invariant sets $K^\top$, $K^\perp$ s.t. for every T-invariant measure μ*

$$\operatorname{supp}\mu \subset K^\top \iff \lambda_1(\mu) = \lambda_1^\top, \qquad \operatorname{supp}\mu \subset K^\perp \iff \lambda_1(\mu) = \lambda_1^\perp$$

(In particular, there exist Lyapunov-optimizing measures.)

Proof:

$$K^\top := \left\{ \xi \in X;\ u \in E_1(\xi),\ n \in \mathbb{Z} \ \Rightarrow\ |||A^{(n)}(\xi)(u)||| = e^{n\lambda_1^\top} |||u||| \right\}.$$

Rem.: The proposition above probably also follows from the general (commutative) theory, but we will use Barabanov functions for other purposes anyway.

OUR MAIN RESULT

Non-overlapping condition

We say that the matrices $A_1, \dots, A_k$ satisfy *forward nonoverlapping condition* (NOC^+) if they admit a multicone M such that

$$i \neq j \;\Rightarrow\; A_i(M) \cap A_j(M) = \emptyset.$$

We say that the matrices $A_1, \dots, A_k$ satisfy *backwards nonoverlapping condition* (NOC^-) if $\{A_1^{-1}, \dots, A_k^{-1}\}$ satisfy the forward nonoverlapping condition.

If both conditions hold then we say that the matrices satisfy the *nonoverlapping condition* (NOC).

Examples: The two previous ones.

The main theorem

Theorem (B., Rams)

*If an one-step cocycle is **dominated** and satisfies the **non-overlapping** condition then the sets $K^\top$, $K^\perp$ have **zero topological entropy**.*

In particular, the optimizing measures have zero entropy.

Remark

- There are examples satisfying the hypotheses of the theorem with no **periodic** λ_1-maximizing measures: Bousch-Mairesse (for some values of the parameters) and others.
- The non-overlapping condition is indeed required: for example, if $A_1 = A_2$ then there are optimal measures with positive entropy.
- We think that a more general theorem holds, with $T =$ subshift of finite type, $A =$ locally constant cocycle, but we haven't checked the details.

Questions

- Can we replace the non-overlapping condition by a weaker hypothesis (preferably "typical" among dominated cocycles)?
- Is there a result of this kind for cocycles that are **not** locally constant?
- What about the **non-dominated** case? (Maybe the restriction of the cocycle to $K^\top$ is typically dominated. . .)
- "Generic finiteness conjecture": Is is typical for Lyapunov-optimizing measures to be **periodic**? (☞ Don't miss Gonzalo Contreras lecture.)
- What about **higher dimension**?

There is much more to be done in this subject!

PROOF OF THE THEOREM

Oseledets directions

Given a bi-infinite word $\xi \in X = k^{\mathbb{Z}}$, split it as $\xi = (\xi_-, \xi_+) \in k^{\mathbb{Z}_-} \times k^{\mathbb{Z}_+}$, where $\mathbb{Z}_- = \{\dots, -2, -1\}$ y $\mathbb{Z}_+ = \{0, 1, \dots\}$.

The first Oseledets direction only depends on the past, while the second one only depends on the future:

$$E_1(\xi) = E_1(\xi_-), \quad E_2(\xi) = E_2(\xi_+).$$

Sets of directions

Consider the sets of Oseledets directions:

$$C_1 := \left\{E_1(\xi_-);\ \xi \in k^{\mathbb{Z}_-}\right\}, \quad C_2 := \left\{E_2(\xi_+);\ \xi \in k^{\mathbb{Z}_+}\right\}.$$

If M is a multi-cone then:

$$C_1 = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\dots,i_n} A_{i_1} \cdots A_{i_n}(M) \quad \text{(nested intersection).}$$

Proof: $E_1(\xi_-)$ appears when we take $i_j = \xi_{-j}$.

If the **forward non-overlapping condition** (NOC^+) is satisfied then C_1 **is a Cantor set**. Moreover, $E_1 : k^{\mathbb{Z}_-} \to C_1$ **is a bijection**.

Similar facts hold for E_2 (consider the inverse multicone $M^{-1} := \mathbb{P}^1 \setminus \overline{M}$.)

Main proposition

Proposition

Suppose that the cocycle satisfies the forward non-overlapping condition (NOC^+).
Let μ be an ergodic measure such that $\lambda_1(\mu) = \lambda_1^\top$ or $\lambda_1(\mu) = \lambda_1^\perp$.
Then for μ-a.e. $\xi \in X$, the direction $E_1(\xi_-)$ is uniquely determined the direction $E_2(\xi_+)$.

Main Proposition $\Rightarrow$ Theorem

Suppose that the cocycle satisfies the non-overlapping condition ($\mathrm{NOC}^+ + \mathrm{NOC}^-$). Let μ be an ergodic measure supported in $K^\top$, that is, $\lambda_1(\mu) = \lambda_1^\top$.

Then **for μ-a.e. $\xi \in X$, the past ξ_- uniquely the future ξ_+**:

$$\begin{tikzcd}
\xi_- \arrow[d] & & \xi_+ \\
E_1(\xi_-) \arrow[rr, "\text{Proposition}", "\mathrm{NOC}^+"'] & & E_2(\xi_+) \arrow[u, "\mathrm{NOC}^-"']
\end{tikzcd}$$

This implies that $h(\mu) = 0$, for all μ supported in $K^\top$.

By the Entropy Variational Principle, $h_{\mathrm{top}}(K^\top) = 0$. Analogously for $K^\perp$.

This proves the Theorem modulo the Proposition.

PROOF OF THE MAIN PROPOSITION

Recall the statement:

Proposition

Suppose that the cocycle satisfies the forward non-overlapping condition (NOC$^+$).
Let μ be an ergodic measure such that $\lambda_1(\mu) = \lambda_1^\top$ or $\lambda_1(\mu) = \lambda_1^\perp$.
Then for μ-a.e. $\xi \in X$, the direction $E_1(\xi_-)$ uniquely determined the direction $E_2(\xi_+)$.

Cross ratio

Given nonzero vectors u, v, u', $v' \in \mathbb{R}^2$, (no three of them being collinear), we define their *cross ratio* (*cross ratio*)

$$[u, v; u', v'] := \frac{u \times u'}{u \times v'} \cdot \frac{v \times v'}{v \times u'} \in \mathbb{R} \cup \{\infty\},$$

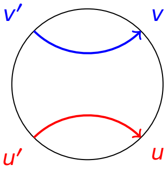
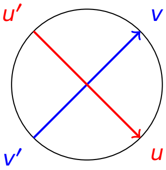
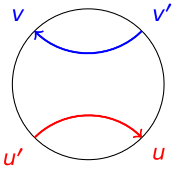
where $\times$ is the cross product on $\mathbb{R}^2$ (that is, determinant).

Actually the value above only depends on the directions determined by the vectors. So we can define the cross ratio of four points in $\mathbb{P}^1$.

The cross ratio is invariant by linear isomorphisms.

Configurations of four points in $\mathbb{P}^1$

Let u, v, u', $v' \in \mathbb{P}^1$ be distinct. The configuration is called *co-parallel*, *crossing* or *anti-parallel* according to the value of the cross ratio $[u, v; u', v']$ as follows:

Co-parallel	**Crossing**	**Anti-parallel**
$0 < [u, v; u', v'] < 1$	$[u, v; u', v'] > 1$	$[u, v; u', v'] < 0$
		

Restrictions among the Oseledets directions

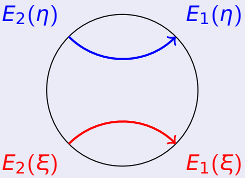
Lemma (Geometrical Lemma)

$K^{\top}$, $K^{\perp}$ = *Mather sets of an* 1*-step dominated cocycle.*

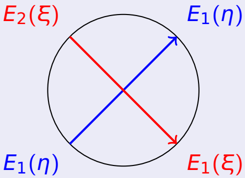
$$\textit{Then}\quad \xi,\eta\in K^{\top} \quad\Rightarrow\quad \big|[E_1(\xi),E_1(\eta);E_2(\xi),E_2(\eta)]\big| \geq 1\,,$$
$$\xi,\eta\in K^{\perp} \quad\Rightarrow\quad \big|[E_1(\xi),E_1(\eta);E_2(\xi),E_2(\eta)]\big| \leq 1\,.$$

In particular:

*co-parallel configuration is **forbidden** en $K^{\top}$:*



*crossing configuration is **forbidden** en $K^{\perp}$:*

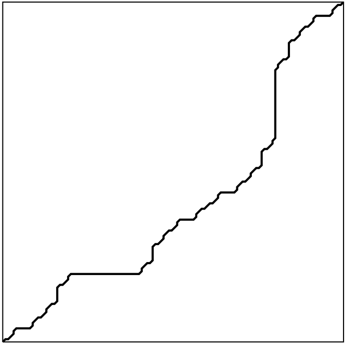


Geometrical Lemma $\Rightarrow$ Main Proposition

Consider the set $G := \left\{(E_1(\xi), E_2(\xi));\ \xi \in K^\top\right\} \subset \mathbb{P}^1 \times \mathbb{P}^1$.

Since the co-parallel configuration is forbidden, the set G has a **monotonicity property**: if E_1 moves clockwise then so does E_2.

This implies that G is a graph (above its projection), with the exception to an at most countable numbers of “plateaux” and “cliffs”:



Geometrical Lemma ⇒ Main Proposition (continued)

Let μ be an ergodic non-periodic (and so non-atomic) measure supported in $K^\top$. Then the union of plateaux and cliffs in G is the image under (E_1, E_2) of a set of zero μ-measure.

Therefore each of the directions E_1 y E_2 uniquely determines the other μ-a.e.

The case of periodic μ being trivial, the Main Proposition is proved in the case of $K^\top$.

The case of $K^\perp$ is similar (but extra care is needed: monotonicity is only local).

PROOF OF THE GEOMETRICAL LEMMA

Some important ideas in the proof come from the 2002 paper by Bousch and Mairesse (counter-examples to the finiteness conjecture).

Recall

Lemma (Geometrical Lemma)

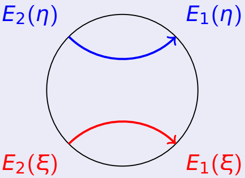
$K^{\top}$, $K^{\perp}$ = Mather sets of an 1-step dominated cocycle.

Then $\xi, \eta \in K^{\top} \quad \Rightarrow \quad \big|[E_1(\xi), E_1(\eta); E_2(\xi), E_2(\eta)]\big| \geq 1\,,$

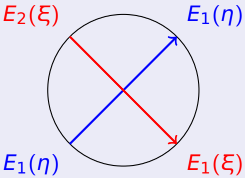
$\xi, \eta \in K^{\perp} \quad \Rightarrow \quad \big|[E_1(\xi), E_1(\eta); E_2(\xi), E_2(\eta)]\big| \leq 1\,.$

In particular:

*co-parallel configuration is **forbidden** en $K^{\top}$:*



*crossing configuration is **forbidden** en $K^{\perp}$:*



An important estimate

Lemma

Let $\xi \in K^\top$, $u \in E_1(\xi)$. Let $v \in \overrightarrow{M}$ be such that $u - v \in E_2(\xi)$. Then $|||u||| \le |||v|||$.

Proof: For all $n \ge 0$,

$$u_n := A^{(n)}(\xi)(u) \quad \Rightarrow \quad |||u_n||| = e^{n\lambda_1^\top} |||u|||$$
$$v_n := A^{(n)}(\xi)(v) \quad \Rightarrow \quad |||v_n||| \le e^{n\lambda_1^\top} |||v|||$$

and so:

$$\frac{|||v|||}{|||u|||} \ge \frac{|||v_n|||}{|||u_n|||} =: \star_n$$

We will show that $\lim_{n\to+\infty} \star_n = 1$.

An important estimate (continued)

Let $|\cdot|$ be the euclidean norm.

$$\begin{aligned} \log |||u_n||| &= \log |||{}^{u_n}\!/_{|u_n|}||| + \log |u_n| \\ \log |||v_n||| &= \log |||{}^{u_n}\!/_{|v_n|}||| + \log |v_n| \end{aligned}$$

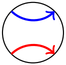
$$\begin{aligned} \left|\log \star_n\right| &= \left|\log |||u_n||| - \log |||v_n|||\right| \\ &\leq \underbrace{\left|\log |||{}^{u_n}\!/_{|u_n|}||| - \log |||{}^{v_n}\!/_{|v_n|}|||\right|}_{(1)} + \underbrace{\left|\log |u_n| - \log |v_n|\right|}_{(2)} . \end{aligned}$$

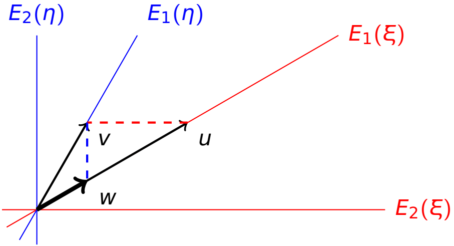
An important estimate (continued)

$$
\begin{aligned}
(1) &= \Big| \log \| u_n/|u_n| \| - \log \| v_n/|v_n| \| \Big| \\
&\le \text{Const} \cdot \angle(u_n, v_n) \\
&\to 0\,, \\
(2) &= \Big| \log |u_n| - \log |v_n| \Big| \\
&\le \max\left(\frac{|u_n|}{|v_n|} - 1, \frac{|v_n|}{|u_n|} - 1 \right) \\
&\le \frac{|u_n - v_n|}{\min(|u_n|, |v_n|)} \\
&\to 0\,.
\end{aligned}
$$

This proves the “important estimate” Lemma.

Proof of the Geometrical Lemma

Assume for a contradiction that ξ, $\eta \in K^\top$ are on co-parallel configuration: . Choose a nonzero vector $u \in E_1(\xi)$, and let v, w be as in the figure.



Apply the “important estimate” twice:

$$|||u||| \le |||v||| \le |||w|||. \quad \text{Contradiction!}$$