# Optimization of Lyapunov Exponents

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# Part 1

# Commutative ergodic optimization: Birkhoff averages

**References:** Surveys by O. Jenkinson.

- *Ergodic Optimization*, Discrete and Cont. Dyn. Sys. A, vol. 15 (2006), pp. 197–224.
- *Ergodic Optimization in Dynamical Systems*, Ergodic Theory Dynam. Systems (2018; online)

**Apology / Disclaimer:** I won't discuss relations with Lagrangian Mechanics, nor Thermodynamical Formalism.

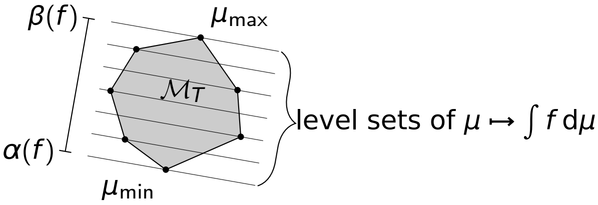
# General setting for the whole talk

- $X =$ compact metric space
- $T : X \to X$ continuous map
- $\mathcal{M}_T :=$ set of $T$-invariant Borel probability measures (compact convex)
- $\mathcal{M}_T^{\text{erg}} :=$ subset of ergodic measures $= \text{ext}(\mathcal{M}_T)$.

# Ergodic optimization of Birkhoff averages

Given a continuous function $f\colon X \to \mathbb{R}$ ("potential"),

$$\left\{ \int f \, \mathrm{d}\mu \; ; \; \mu \in \mathcal{M}_T \right\} =: [\alpha(f), \beta(f)]$$



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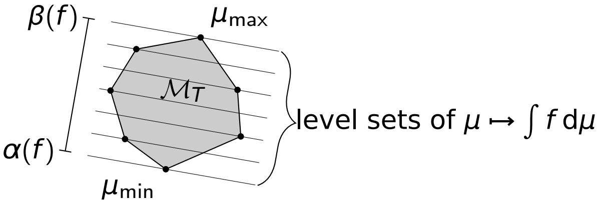


$\mu \in \mathcal{M}_T$ s.t. $\int f \, \mathrm{d}\mu = \beta(f)$ is called a **maximizing measure**.

Note: **Ergodic** maximizing measures always exist. In particular, uniqueness ⇒ ergodicity.

# Expressing $\beta(f)$ in terms of Birkhoff averages

Birkhoff sum $f^{(n)} := f + f \circ T + \cdots + f \circ T^{n-1}$

$$\beta(f) = \sup_{x \in X} \limsup_{n \to \infty} \frac{f^{(n)}(x)}{n}$$
$$= \lim_{n \to \infty} \sup_{x \in X} \frac{f^{(n)}(x)}{n}$$

# Ergodic optimization of Birkhoff averages

**Meta-Problem**

*To understand maximizing measures.*

# Maximizing measures: Generic uniqueness

**Theorem (Conze–Guivarch, Jenkinson, . . . )**

*Let $\mathcal{F}$ be any "reasonable" space $\mathcal{F}$ of continuous functions.*
*For generic $f$ in $\mathcal{F}$, the maximizing measure is* ***unique****.*

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*For generic $f$ in $\mathcal{F}$, the maximizing measure is **unique**.*

"Reasonable" space: a topological vector space $\mathcal{F}$ continuously and densely embedded in $C^0(X)$.

Generic property: a property that holds on a dense $G_\delta$ subset (of a Baire space).

# The inverse problem

**Theorem (Jenkinson)**

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If $\mu$ has finite support then $f$ can be taken $C^\infty$.

For a general $\mu$, how regular $f$ can be taken? Not much. . .

# Maximizing sets

Assume the following **nice setting**:

- $T: X \to X$ is "**hyperbolic**" (e.g. uniformly expanding, Anosov);
- $f: X \to \mathbb{R}$ is "**regular**" (at least Hölder).

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- It is **false** if $f$ is only $C^0$ (by the previous theorem)
- It is a corollary of the **Mañé Lemma** (or **Revelation Lemma** or **Nonpositive Livsic Lemma**).
  Several formulations: Mañé'92, Conze–Guivarc'h'93, Fathi'97, Savchenko'99, Bousch'00, Contreras–Lopes–Thieullen'01, Lopes–Thieullen'03, Pollicott–Sharp'04, Bousch'11).

# Expected panorama for the nice setting

**Meta-Conjecture (∼ Hunt–Ott, Phys. Rev. 1996)**

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*Then for typical*
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# Known results in this direction

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**Theorem (Contreras'16)**

*$T$ unif. expanding ⇒ for generic Lipschitz $f$'s (actually all $f$'s in an* ***open*** *and dense subset), the maximizing measure is supported on a periodic orbit.*

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Only result with a probabilistic notion of typicality (**prevalence**): B., Zhang'16.

# A beautiful example

Conze, Guivarch'93; Hunt–Ott'96; Jenkinson'96; Bousch'00

$T(x) = 2x \bmod 2\pi$ on the circle $X := \mathbb{R}/2\pi\mathbb{Z}$

$f =$ trigonometric polynomial of deg. 1
WLOG, $f(x) = f_\theta(x) = \cos(x - \theta)$

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## Theorem (Bousch'00)

*For every $\theta \in [0, 2\pi]$, the function $f_\theta$ has a unique maximizing measure $\mu_\theta$, and it has zero entropy (actually, Sturmian).*

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**Theorem (Bousch'00)**

*For every $\theta \in [0, 2\pi]$, the function $f_\theta$ has a unique maximizing measure $\mu_\theta$, and it has zero entropy (actually, Sturmian).*
*Furthermore, for Lebesgue-a.e. $\theta$ (actually, all $\theta$ outside a set of Hausdorff dim. 0), $\mu_\theta$ is supported on a periodic orbit.*

# Part 2

# Non-commutative ergodic optimization: Top Lyapunov exponent

Replace the scalar function $f$ by a (continuous) matrix-valued function:

$$F\colon X \to \mathrm{Mat}(d \times d, \mathbb{R}) \text{ or } \mathrm{GL}(d, \mathbb{R}) \qquad (\text{“cocycle”}).$$

The Birkhoff sums are replaced by products:

$$F^{(n)}(x) := F(T^{n-1}x) \cdots F(Tx)F(x)\,.$$

**Top Lyapunov exponent**:

$$\lambda_1(F, x) := \lim_{n\to\infty} \frac{1}{n} \log \|F^{(n)}(x)\| \quad \text{(if it exists)}$$

For any $\mu \in \mathcal{M}_T$, the limit exists for $\mu$-a.e. $x \in X$.

$$\lambda_1(F, \mu) := \int \lambda_1(F, x)\, \mathrm{d}\mu(x)$$

# Optimization of the top Lyapunov exponent

Quantities of interest:

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu)$$

$$\beta(F) := \sup_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu)$$

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For “step cocycles”:

- $e^{\beta(F)}$ is called **joint spectral radius** (Rota, Strang’60; Daubechies, Lagarias’92, ...)
- $e^{\alpha(F)}$ is called **joint spectral subradius** (Gurvits’95).

Another characterization:

$$\beta(F) = \lim_{n\to\infty} \sup_{x\in X} \frac{1}{n} \log \|F^{(n)}(x)\| .$$

# $\lambda_1$-minimizing/maximizing measures?

**Basic difficulty:**
$\mu \in \mathcal{M}_T \mapsto \lambda_1(F, \mu)$ is **not continuous**, in general.
It is **upper semi-continuous**, at least.

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It is **upper semi-continuous**, at least.

$$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu)$$ ☹ not necessarily attained

$$\beta(F) := \sup_{\mu \in \mathcal{M}_T} \lambda_1(F, \mu)$$ ☺ always attained

# Example without $\lambda_1$-minimizing measure

Step cocycle $T\colon \{0,1\}^{\mathbb{N}} \hookleftarrow$ shift, $F(x) = A_{x_0}$ where $A_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1/8 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

## Claim

$\alpha(F) := \inf_{\mu \in \mathcal{M}_T} \lambda_1(F,\mu) = -\log 2$, but the inf is not attained.

## Proof.

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$$A_1 A_0^n = \begin{pmatrix} 0 & -2^{-3n} \\ 2^n & 0 \end{pmatrix} \text{ has eigenvalues } \pm 2^{-2n} i, \text{ so}$$

$$\boxed{\mu_n := \delta_{(0^n 1)^\infty}} \Rightarrow \lambda_1(F,\mu_n) = \boxed{-\frac{n}{n+1}\log 2} \searrow -\log 2.$$

So $\boxed{\alpha(F) \le -\log 2}$. Discontinuity: $\lambda_1(F, \lim \mu_n) \neq \lim \lambda_1(F, \mu_n)$.

On the other hand. . . □

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## Proof.

$$\lambda_1(F,\mu) \overset{(1)}{\geq} \frac{\lambda_1(F,\mu)+\lambda_2(F,\mu)}{2} = \int \frac{1}{2} \log \underbrace{|\det F(x)|}_{\geq 1/4} \, d\mu(x) \overset{(2)}{\geq} -\log 2 \, .$$

So $\boxed{\alpha(F) \geq -\log 2}$ and therefore $\boxed{\alpha(F) = -\log 2}$.
Moreover, (2) becomes "$=$" iff $\mu = \delta_{0^\infty}$, but then (1) is "$>$". So no $\mu$ attains $\lambda_1(F,\mu) = -\log 2$. □

# Expected panorama for $\lambda_1$-maximization

**Meta-Conjecture**

*Suppose T is chaotic (unif. expanding / unif. hyperbolic / . . . ). Then for typical (topological sense / probabilistic sense) regular (Hölder / . . . / analytic) cocycles F, the $\lambda_1$-maximizing measure has low complexity (zero topological entropy / . . . / supported on a periodic orbit).*

A result that fits this philosophy: B., Rams'16.

# Some initial results

Similarly to the commutative **subordination principle**:

**Theorem (B., Garibaldi)**

*Suppose $T$ is a hyperbolic homeomorphism, and that $F$ is a ~~strongly~~ fiber-bunched cocycle. Then there exists a* ***maximizing set****: a $T$-invariant compact set $K \subseteq X$ such that*

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This is actually a corollary of a **version of Mañé Lemma for cocycles** (existence of extremal norms), which has other applications.

Related work: Morris'10, Morris'13.

# Part 3

# Non-commutative ergodic optimization: Full Lyapunov spectra

**Extra information:** Proceedings paper (ArXiv 1712.01612)

# The other Lyapunov exponents

$T\colon X \to X$, $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ as before.
For each $i \in \{1, 2, \dots, d\}$, and $x \in X$, let

$$\lambda_i(F,x) := \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{s}_i(F^{(n)}(x)) \quad \text{(if it exists)}$$

where $\mathbf{s}_i(\cdot) :=$ $i$-th biggest singular value.

For any $\mu \in \mathcal{M}_T$, these limits exist for $\mu$-a.e. $x \in X$.
If $\mu$ is **ergodic**, then $\lambda_i(F,\cdot)$ is $\mu$-a.e. equal to some constant $\lambda_i(F,\mu)$.

# Lyapunov spectrum of a cocycle

Given $(T, F)$, the **Lyapunov vector** of $\mu \in \mathcal{M}_T^{\text{erg}}$ is:

$$\vec{\lambda}(F,\mu) := \big(\lambda_1(F,\mu), \dots, \lambda_d(F,\mu)\big)$$

The **Lyapunov spectrum** of $(T, F)$ is:

$$L^+(F) := \left\{\vec{\lambda}(F,\mu) \,;\, \mu \in \mathcal{M}_T^{\text{erg}}\right\},$$

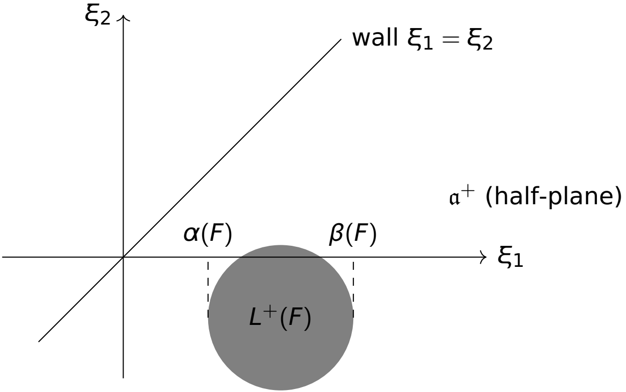
which is a subset of the **positive chamber**:

$$\mathfrak{a}^+ := \left\{(\xi_1, \dots, \xi_d) \in \mathbb{R}^d \,;\, \xi_1 \geq \cdots \geq \xi_d\right\}.$$

# Lyapunov spectrum of a cocycle

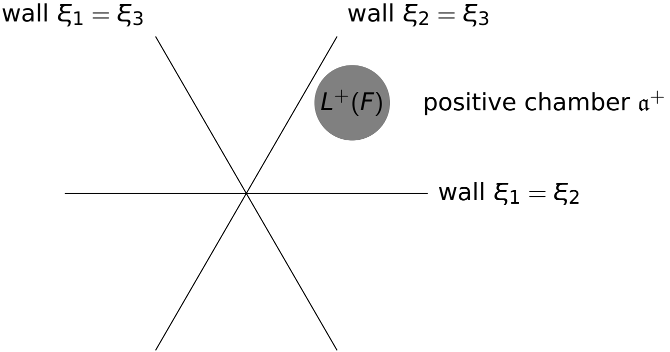
$$L^+(F) := \left\{\vec{\lambda}(F,\mu) \,;\, \mu \in \mathcal{M}_T^{\mathrm{erg}}\right\}$$

$$\subset \quad \mathfrak{a}^+ := \left\{(\xi_1,\dots,\xi_d) \in \mathbb{R}^d \,;\, \xi_1 \ge \dots \ge \xi_d\right\}.$$



If $F$ takes values in $\mathrm{SL}(3,\mathbb{R})$ then the Lyapunov spectrum is also contained in the plane

$$\left\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \; ; \; \xi_1 + \xi_2 + \xi_3 = 0\right\}$$




Related: Sert'17 has a notion of "joint spectrum" (more general Lie groups); he proves large deviation results.

# A nice result (for the "nice setting")

**Theorem (Kalinin'11)**

*Suppose $T\colon X \to X$ is hyperbolic, and $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ is a Hölder-continuous cocycle. Then the Lyapunov vectors of measures supported on* ***periodic orbits*** *are dense in the Lyapunov spectra $L^+(F)$.*

# Expected picture of $L^+(F)$

**Meta-Conjecture (Typical spectra; part 1)**

*Suppose $T\colon X \to X$ is hyperbolic, and $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ is a typical regular cocycle. Then:*

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2. *Its boundary is **"fishy"**.*
3. *Every boundary point $\vec{\xi}$ **outside the walls** is attained as the Lyapunov vector of a unique ergodic measure $\mu_{\vec{\xi}}$; furthermore, $\mu_{\vec{\xi}}$ has low complexity (zero topological entropy).*

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3. *Every boundary point $\vec{\xi}$ **outside the walls** is attained as the Lyapunov vector of a unique ergodic measure $\mu_{\vec{\xi}}$; furthermore, $\mu_{\vec{\xi}}$ has low complexity (zero topological entropy).*
4. *Subordination property: these $\mu_{\vec{\xi}}$ have uniquely ergodic supports.*

# A particular but concrete example

"Step cocycle" $T\colon \{0,1\}^{\mathbb{N}} \hookleftarrow$ shift, $F(x) = A_{x_0}$ where $A_0 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_1 = \left(\begin{smallmatrix} 2 & 0 \\ 2 & 2 \end{smallmatrix}\right)$. Then:

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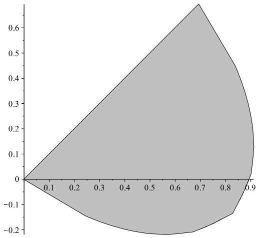
- $L^+(F)$ is convex.
- Its boundary is composed of a piece of the wall $\xi_1 = \xi_2$ and a curve with a **dense subset of corners** – "fishy".
- Every point in this curve is attained as the Lyapunov vector of a unique ergodic measure, which is Sturmian.




Corollary of works by Hare, Morris, Sidorov, Theys'11; Morris, Sidorov'13 (on counterexamples for the "finiteness conjecture"; see also Bousch, Mairesse'01).

# The simplest case

If $F(x) = \begin{pmatrix} e^{f_1(x)} & 0 \\ 0 & e^{f_2(x)} \end{pmatrix}$ where $\boxed{f_1 > f_2}$ then:

- The Lyapunov vector $\mu \mapsto \vec{\lambda}(F, \mu)$ is continuous, since it equals $\int \vec{f} \, d\mu$ where $\vec{f} = (f_1, f_2)$.

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- The Lyapunov spectrum $L^+(F)$ is a "rotation set"; in particular it is compact and convex, and its extremal points are attained by ergodic measures.
- $L^+(F)$ is away from the wall $\xi_1 = \xi_2$.

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- $L^+(F)$ is away from the wall $\xi_1 = \xi_2$.

**Commutativity regained:** Essentially the same happens if the cocycle admits a dominated splitting into one-dimensional bundles – which is an open property.

# A step back: vectorial ergodic optimization

The **rotation set** of a continuous $\vec{f}: X \to \mathbb{R}^d$ is:

$$R(\vec{f}) := \left\{ \int \vec{f} \, \mathrm{d}\mu \, ; \mu \in \mathcal{M}_T \right\}$$

compact and convex; an affine projection of $\mathcal{M}_T$ in $\mathbb{R}^d$.

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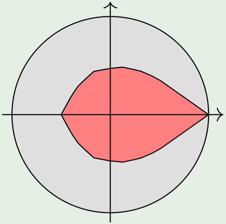
## Example (The fish: Jenkinson'96, Bousch'00)

$T(x) = 2x \bmod 2\pi$, $\vec{f}(x) = (\cos x, \sin x)$.



- Fishy boundary: dense subset of corners.
- Each corner comes from a periodic orbit.
- Boundary points come from low-complexity measures (Sturmian).

**Note:** No Mañé Lemma for vectorial ergodic optimization (B., Delecroix) – see Proceedings paper.

# Back to cocycles: Dominated splittings

Suppose the cocycle $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ admits an **invariant** splitting:

$$\mathbb{R}^d_x = \underbrace{V_x}_{\dim = i} \oplus \underbrace{W_x}_{\dim = d-i} \qquad F(x)(V_x) = V_{Tx}, \ F(x)(V_x) = W_{Tx}\,.$$

The splitting is **dominated** if $\exists c \in (0,1)$ s.t. (changing the norm if necessary)

$$\|F(x)w\| < c\|F(x)v\| \quad \forall x, \ \forall \text{unit vectors } v \in V_x, \ w \in W_x\,.$$

($\Leftrightarrow$ uniform exponential separation of singular values $\mathbf{s}_i$, $\mathbf{s}_{i+1}$ for the products $F^{(n)}(x)$: B., Gourmelon'09)

# Finest dominated splitting

Every cocycle admits a **finest dominated splitting** $\mathbb{R}^d = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ (maybe **trivial** ($k = 1$)).

If the splitting is **simple** ($k = d$) then we recover commutativity.

**Possible strategy to obtain convexity of $L^+(F)$:** use subsystems with simple dominated splitting?

# Domination vs. Lyapunov exponents

If a cocycle admits a dominated splitting with dominating bundle of dim. $i$ then the Lyapunov spectrum $L^+(F)$ is **away from the wall** $\xi_i = \xi_{i+1}$.

The converse is false . . .

# Domination vs. Lyapunov exponents

If a cocycle admits a dominated splitting with dominating bundle of dim. $i$ then the Lyapunov spectrum $L^+(F)$ is **away from the wall** $\xi_i = \xi_{i+1}$.

The converse is false . . . but maybe true for typical cocycles? (known counterexamples are too delicate)

# More questions. . .

## Meta-Conjecture (Typical Lyapunov spectra – continued)

*Suppose $T: X \to X$ is hyperbolic, and $F: X \to \mathrm{GL}(d, \mathbb{R})$ is a typical regular cocycle. Then:*

1. *The Lyapunov spectrum $L^+(F)$ is a **convex** set.*
2. *Its boundary is **"fishy"** (dense subset of corners away from walls).*
3. *Every boundary point $\vec{\xi}$ **outside the walls** is attained as the Lyapunov vector of a unique ergodic measure $\mu_{\vec{\xi}}$; furthermore, $h(\mu_{\vec{\xi}}, T) = 0$.*
4. *Subordination property: these $\mu_{\vec{\xi}}$ have uniquely ergodic supports.*
5. *$L^+(F)$ touches the wall $\xi_i = \xi_{i+1}$ iff there is no dominated splitting with dominating bundle of dim. $i$.*

# Extra convexity properties of $L^+(F)$?

Let's add still another item:

**Meta-Conjecture (Typical Lyapunov spectra – continued)**

*Suppose $T\colon X \to X$ is hyperbolic, and $F\colon X \to \mathrm{GL}(d,\mathbb{R})$ is a typical regular cocycle. Then:*

...

6. *There exists a (larger)* ***convex*** *set $M^+(F) \subset \mathbb{R}^d$ (**Morse set**) such that $\boxed{M^+(F) \cap \mathfrak{a}^+ = L^+(F)}$ and $M^+(F)$ is invariant by reflections across the walls it touches.*

**Remark:** The terminology **Morse set** comes from Control Theory: Colonius, Kliemann'96, '02 – chain transitivity on projective and flag bundles.

# Extra convexity properties of $L^+(F)$?

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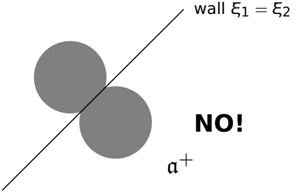


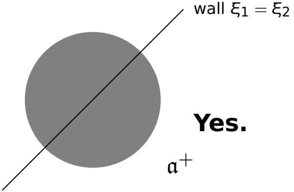

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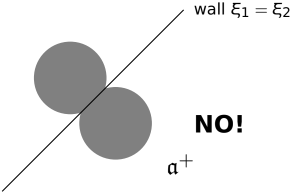


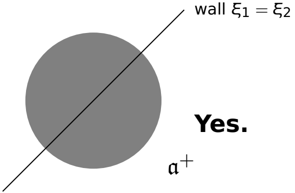


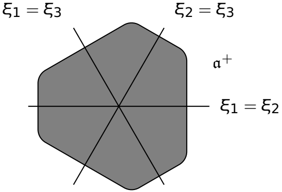


($F$ in $\mathrm{SL}(3, \mathbb{R})$; no dominations)

# Rationale for (6) extra convexity

**Philosophy:** Lack of domination (of “index” *i*) should allow us to mix (make convex combinations) of Lyapunov exponents $\lambda_i$ and $\lambda_{i+1}$.

**Example (seen before)**

The step cocycle induced by matrices $\begin{pmatrix} 2 & 0 \\ 0 & 1/8 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

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Some implementations of this "philosophy": B.'01; B., Viana'05; B., Bonatti'12 (perturbative). Gorodetski, Ilyashenko, Kleptsyn, Nalsky'05; B., Bonatti, Díaz'14, '16 (non-perturbative).

On the other hand, if a conjecture by B., Fayad'06 is true then (6) is false for probabilistic-typical step cocycles in dim. 2. (But step cocycles don't look very typical. . . )