Finiteness of matrix equilibrium states

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Main reference

This talk is based on the paper:

J. B., Ian D. Morris. Equilibrium states of generalised singular value potentials and applications to affine iterated function systems. *Geometric and Functional Analysis*, 28 (2018), no. 4, pp. 995–1028.

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Subadditive thermodynamical formalism

Subadditive pressure

Given:

- a compact metric space X;
- a continuous map $T\colon X \to X$;
- a sequence $\mathcal{F} = (f_n)_{n\geq 1}$ of continuous functions $f_n\colon X \to [-\infty, +\infty)$ which is **subadditive**:

$$f_{n+m} \leq f_n \circ T^m + f_m \,.$$

Define the **(topological) pressure**:

$$P(\mathcal{F}) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sup_{\substack{E\subseteq X \\ (n,\varepsilon)\text{-separated}}} \frac{1}{n} \log \sum_{x\in E} e^{f_n(x)} \,.$$

Subadditive ergodic theorem

$\mathcal{M}_T(X) := \{T\text{-invariant probability measures}\}.$

A Borel set $B \subseteq X$ has **full probability** if

$$\mu(B) = 1, \quad \forall \mu \in \mathcal{M}_T(X).$$

Kingman'68: If $\mathcal{F} = (f_n)$ is a (say, continuous) subadd. seq. then the **asymptotic average**

$$\bar{f}(x) := \lim_{n\to\infty} \frac{f_n(x)}{n} \quad \text{exists for all } x \text{ in a full probability set.}$$

Furthermore, for all $\mu \in \mathcal{M}_T$,

$$\int \bar{f}\, d\mu = \underbrace{\lim_{n\to\infty}}_{\inf} \int \frac{f_n}{n}\, d\mu\,.$$

Subadditive variational principle

Subadditive variational principle: If $\bar{f}$ is the asymptotic avg. of the subadd. seq. $\mathcal{F}$, then:

$$P(\mathcal{F}) = \sup_{\mu \in \mathcal{M}_T(X)} \left(h_\mu(T) + \int \bar{f}\, d\mu \right) \qquad \text{(if } h_{\text{top}}(T) < \infty\text{)}.$$

(Cao–Feng–Huang'08; related work by Falconer'88, Barreira'96, Kaënmäki'04, Mummert'06.)

An **equilibrium state** is an invariant measure μ that attains the sup.

As in the classical (additive) setting, (ergodic) equilibrium states exist provided the metric entropy is upper-semicontinuous (e.g. T expansive or C^∞).

An easy remark: If (f_n) and (g_n) are two subadditive sequences, then we can construct a third one by:

$$h_n(x) := \max\left\{f_n(x), g_n(x)\right\}.$$

Asymptotic average:

$$\bar{h}(x) = \max\left\{\bar{f}(x), \bar{g}(x)\right\}.$$

If μ is ergodic, then

$$\int \bar{h}\,d\mu = \max\left\{\int \bar{f}\,d\mu,\ \int \bar{g}\,d\mu\right\}.$$

So:

$$\{\text{erg. equil. states for } (h_n)\} \subseteq \{\text{erg. equil. states for } (f_n)\} \cup \{\text{erg. equil. states for } (g_n)\}.$$

Locally constant case

Consider:

- $(X,T) = (\Sigma_N, \sigma) =$ one-sided full shift on the alphabet $\{1,\dots,N\}$.
- a subadditive sequence $\mathcal{F} = (f_n)$ s.t. each f_n is constant on the cylinders of depth n.

Equivalently, for every word w

$$f_n|_{[w]} \equiv \log \Phi(w),$$

where $\Phi\colon \Sigma_N^* \to [0,+\infty)$ is a **submultiplicative potential**, i.e., a function on the set Σ_N^* of words s.t.

$$\forall w, v \in \Sigma_N^*, \quad \Phi(wv) \leq \Phi(w)\Phi(v)\,.$$

Rem.: Even if the subadd. seq. (f_n) (for the shift) is not loc. const., we can still define a submult. potential $\Phi(w) := \exp \sup_{[w]} f_{|w|}$.

In the locally constant situation $f_n|_{[w]} = \log \Phi(w)$, the pressure has a simpler expression:

$$P(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\substack{w\in\Sigma_N^* \\ |w|=n}} \Phi(w)$$

That is,

$$\sum_{\substack{w\in\Sigma_N^* \\ |w|=n}} \Phi(w) = e^{nP(\Phi)+o(n)}$$

Semi- and Quasi-multiplicativity

Let $\Phi\colon \Sigma_N^* \to [0,+\infty)$ be a submultiplicative potential.

- Φ is **semimultiplicative** if there exists $c \in (0,1]$ such that for every pair of words w, v,

  $$\Phi(wv) \geq c\Phi(w)\Phi(v)\,.$$

  Example: Norm potential under 1-domination hypothesis.

- Φ is **quasimultiplicative** if there exists $c \in (0,1]$ and $\ell \in \mathbb{N}$ such that for every pair of words w, v there exists a word u of length $|u| \leq \ell$ such that:

  $$\Phi(wuv) \geq c\Phi(w)\Phi(v)\,.$$

  Example: Norm potential under irreducibility hypothesis (more about this later).

Consequences of quasimultiplicativity

Theorem (Feng'11)

Every quasimultiplicative potential Φ has a ***unique*** *equilibrium state μ, which is ergodic, and satisfies* ***Gibbs inequalities****: there exists $C > 0$ such that for every cylinder $[w] \subseteq \Sigma_N$,*

$$C^{-1}\,\Phi(w)\,e^{-|w|P(\Phi)} \leq \mu([w]) \leq C\,\Phi(w)\,e^{-|w|P(\Phi)}\,.$$

In particular, μ has full support.

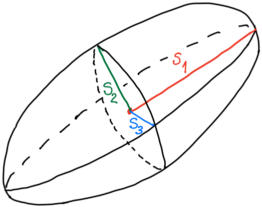
Singular values, Lyapunov exponents

Singular values

A linear map $L\colon \mathbb{R}^d \to \mathbb{R}^d$ has **singular values**

$$\mathfrak{s}_1(L) \geq \cdots \geq \mathfrak{s}_d(L).$$

So $\mathfrak{s}_1(L) = \|L\|$, $\mathfrak{s}_d(L) =$ "co-norm".



Ellipsoid $L(S^{d-1})$.

Exterior powers: $\Lambda^k \mathbb{R}^d = \mathbb{R}^{\binom{d}{k}}$, $\Lambda^k L\colon \Lambda^k\mathbb{R}^d \to \Lambda^k\mathbb{R}^d$.

$$\begin{aligned}\|\Lambda^k(L)\| &= \text{biggest expansion rate of } k\text{-volume} \\ &= \mathfrak{s}_1(L)\cdots\mathfrak{s}_k(L)\,.\end{aligned}$$

Submultiplicativity: $\|\Lambda^k(L_1L_2)\| \leq \|\Lambda^k(L_1)\|\,\|\Lambda^k(L_2)\|$.

Lyapunov exponents

Given a continuous **matrix cocycle** $A\colon X \to \mathrm{Mat}(d \times d)$, we form the products:

$$A^{(n)}(x) := A(T^{n-1}x) \cdots A(Tx)A(x)\,.$$

For all x on a full probability set, the **Lyapunov exponents**

$$\lambda_1(x) \geq \cdots \geq \lambda_d(x), \qquad \lambda_k(x) := \lim_{n\to\infty} \frac{1}{n} \log \mathfrak{s}_k(A^{(n)}(x))$$

exist. Indeed, for each k, $\lambda_1 + \cdots + \lambda_k$ is the asymptotic average of the following subadditive sequence:

$$f_{n,k}(x) := \log \|\Lambda^k(A^{(n)}(x))\| = \sum_{i=1}^{k} \log \mathfrak{s}_i(A^{(n)}(x))\,.$$

Singular value potential

Given an ordered vector

$$\vec{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \quad \text{with} \quad \alpha_1 \geq \cdots \geq \alpha_d,$$

the function

$$\varphi_{\vec{\alpha}} \colon \mathrm{Mat}(d \times d) \to [0, +\infty), \quad \varphi_{\vec{\alpha}}(L) := \prod_{i=1}^{d} \mathfrak{s}_i(L)^{\alpha_i}$$

is submultiplicative: $\varphi_{\vec{\alpha}}(L_1 L_2) \leq \varphi_{\vec{\alpha}}(L_1)\varphi_{\vec{\alpha}}(L_2)$. Indeed:

$$\varphi_{\vec{\alpha}}(L) = \left(\prod_{i=1}^{d-1} \underbrace{\|\Lambda^i L\|^{\alpha_i - \alpha_{i+1}}}_{\text{submult.}} \right) \underbrace{\|\Lambda^d L\|^{\alpha_d}}_{\text{multiplicative}}$$

Singular value pressure

Given the cocycle (T, A) and a ordered vector $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d_{\downarrow}$ consider the subadditive sequence

$$f_{n,\vec{\alpha}}(x) := \log \varphi_{\vec{\alpha}}(A^{(n)}(x)), \quad \text{where} \quad \varphi_{\vec{\alpha}} := \prod_{i=1}^{d} \mathfrak{s}_i^{\alpha_i}$$

The corresponding pressure is:

$$P(A, \vec{\alpha}) = \sup_{\mu \in \mathcal{M}_T(X)} \left(h_\mu(T) + \int (\alpha_1 \lambda_1 + \cdots + \alpha_d \lambda_d) \, d\mu \right)$$

(provided $h_{\text{top}}(T) < \infty$).

Some subjects that we won't go into:

- Continuity of the singular value pressure: See Feng–Shmerkin'14, Morris'16, Cao–Pesin–Zhao'19.
- Differentiability of the singular value pressure.
- Multifractal analysis of Lyapunov exponents: Given a linear cocycle, how big (in terms of topological entropy) is the set of points with a given Lyapunov spectrum? See Feng–Huang'10.
- Transfer operators and applications: See Guivarc'h–LePage'04, Piraino'20.

Applications to dimension theory

Contracting IFS

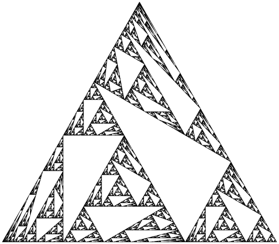
Consider an IFS (iterated function system) specified by contractions $T_1, \ldots, T_N \colon \mathbb{R}^d \to \mathbb{R}^d$. Its **attractor** is the unique nonempty compact set $\Lambda \subseteq \mathbb{R}^d$ such that

$$\Lambda = \bigcup_{i=1}^{N} T_i(\Lambda) \,.$$

In fact, Λ consists of all limits

$$\lim_{i\to\infty} T_{i_n} \circ \cdots \circ T_{i_1}(x).$$

If the contractions T_i are affine then Λ is called a **self-affine set**.



A non-conformal Sierpiński gasket.

Dimension estimate

Let $\Lambda \subseteq \mathbb{R}^d$ be a self-affine set. Let A be the loc. const. cocycle induced by the linear parts of the contractions. The unique root s of the “Bowen-like” equation

$$\boxed{P(A, \vec{\alpha}(s)) = 0} \quad \text{where } \vec{\alpha}(s) := \big(\underbrace{1, \ldots, 1}_{\lfloor s \rfloor}, s - \lfloor s \rfloor, 0, \ldots, 0\big)$$

is called **affinity dimension** of the IFS.

Theorem (Falconer’88)

$\dim_H(\Lambda) \leq \dim_{aff}(T_1, \ldots, T_N)$.

Rem.: A corresponding equilibrium states on Σ_N project to measures on Λ which are natural candidates for measure of maximal dimension (Käenmäki’04).

Dimension of typical fractals

Contractions $T_i(x) = A_i(x) + b_i \rightsquigarrow$ self affine-set $\Lambda \subseteq \mathbb{R}^d$.

Falconer bound: $\dim_{\mathrm{H}}(\Lambda) \leq \dim_{\mathrm{aff}}(T_1, \ldots, T_N)$. $(\star)$

Theorem (Falconer'88, Solomyak'98)

Equality holds in $(\star)$ for sufficiently strong contractions ($\|A_i\| < 1/2$) and Lebesgue-a.e. displacements (b_i).

Theorem (Bárány–Hochman–Rapaport'19)

Equality holds in $(\star)$ if:

- $d = 2$,
- *$T_1(\Lambda), \ldots, T_N(\Lambda)$ pairwise disjoint (**strong separation**),*
- *the linear parts $A_1, \ldots, A_N$ admit no common invariant set of lines (**strong irreducibility**) nor a common invariant conformal structure.*

Nonlinear fractals

Theorem (Ban–Cao–Hu'10 (after Zhang'97, Barreira'03))

Given a repeller Λ for a C^1 expanding map f, consider the cocycle $(T, A) = (f|_\Lambda, Df|_\Lambda)$. Then the unique root of the "Bowen-like" equation

$$\boxed{P(A, \vec{\alpha}_s^*) = 0} \quad \text{where } \vec{\alpha}_s^* := \Big(0, \ldots, 0, -s + \lfloor s \rfloor, \underbrace{-1, \ldots, -1}_{\lfloor s \rfloor}\Big)$$

gives an upper bound for the Hausdorff dimension of Λ.

Question (Difficult, I suppose)

Is this upper bound typically sharp (among $f \in C^{1+\theta}$, say)?

Uniqueness or finiteness of equilibrium states for the singular value pressure

Example of nonunique EES

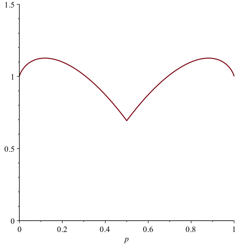
Consider the following pair of 2×2 matrices:

$$A_1 := \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$$

Claim: There are two ergodic equilibrium states for the norm potential $\Phi = \|\cdot\|$.

Indeed, since the matrices commute, the candidates for equilibrium states are Bernoulli measures μ_p, $p \in [0, 1]$.

$$h_{\mu_p}(\sigma) + \lambda_1(A, \mu_p) = \\ -p \log p - (1-p)\log(1-p) \\ + |1-2p| \log 2 =$$



Irreducibility ⇒ unique EES for the norm potentials

A tuple $(A_1, \ldots, A_N)$ of $d \times d$ matrices is **irreducible** if there is no nontrivial common invariant subspace. This property only depends on the generated semigroup

$$S := \langle A_1, \ldots, A_N \rangle \subseteq \mathrm{Mat}(d \times d).$$

Theorem (Feng'09)

If $(A_1, \ldots, A_N)$ is irreducible then the norm potential is quasimultiplicative:

$$\exists c > 0 \ \exists \ell > 0 \ \forall B, C \in S \ \exists M \in S \textit{ with } \mathrm{length}(M) \leq \ell \quad \textit{s.t.} \quad \|BMC\| \geq c\|B\| \, \|C\| \, .$$

In particular, $\forall \alpha > 0$, the submult. potential $\Phi := \|\cdot\|^\alpha$ has a unique equil. state (and it has the Gibbs property).

Proof.

By contradiction, assume that there exist sequences $\varepsilon_n \to 0$, $B_n \in S$, $C_n \in S$ such that:

$$\forall M \in S, \quad \text{length}(M) \le n \quad \Rightarrow \quad \frac{\|B_n M C_n\|}{\|B_n\|\,\|C_n\|} < \varepsilon_n\,.$$

Passing to subsequences, $\frac{B_n}{\|B_n\|} \to B$ and $\frac{C_n}{\|C_n\|} \to C$. Then:

$$\forall M \in S, \quad BMC = 0.$$

That is,

$$S(\text{Im}(C)) = \bigcup_{M \in S} M(\text{Im}(C)) \subseteq \text{Ker}(B)\,.$$

Since $\|B\| = \|C\| = 1$, $\text{Im}(C) \ne \{0\}$ and $\text{Ker}(B) \ne \mathbb{R}^d$. So $S(\text{Im}(C))$ spans a proper S-invariant subspace. □

Finiteness of EES for the norm potentials

Theorem (Feng–Käenmäki?)

Let $(A_1, \ldots, A_N)$ be any tuple of $d \times d$ matrices. Then, for every $\alpha > 0$, the submultiplicative potential $\Phi := \|\cdot\|^\alpha$ admits at most d ergodic equilibrium states.

Proof.

- If the tuple is irreducible, then Φ is quasimultiplicative.
- If the tuple is reducible then write it in block-triangular form, apply the previous result to each diagonal block.

□

Typical uniqueness of EES for sing. val. potentials

Theorem (Järvenpää–Järvenpää–Li–Stenflo'16)

For typical tuples of $d \times d$ matrices[a], the singular value potentials are quasimultiplicative, and in particular equilibrium states are unique and fully supported.

[a] in the complement of an algebraic subset of positive codimension

Theorem (Park'20)

For typical fiber-bunched Hölder cocycles[a], the singular value potentials are quasimultiplicative, and in particular equilibrium states are unique and fully supported.

[a] satisfying pinching & twisting conditions a la Bonatti–Viana, Avila–Viana

Finiteness of EES for sing. val. potentials

The following answers a question of Käenmäki'04:

Theorem (B.–Morris'18)

Take any tuple of invertible $d \times d$ matrices. Then every singular value potential admits finitely many ergodic equilibrium states, and all of them are fully supported.

Previous results: Feng–Käenmäki ($d = 2$), Käenmäki–Morris ($d = 3$), Käenmäki–Li (some $\vec{\alpha} \in \mathbb{Q}^d_{\downarrow}$).

Remarks:

- The bound on the number of EES depends only on d.
- Should work for locally constant cocycles over SFT (or sofic shifts). . .

Curious corollary

Corollary (previously a folklore open question)

If $N \geq 2$ and $T_1, \ldots, T_N$ are invertible affine contractions, then

$$\dim_{\mathrm{aff}}(T_1, \ldots, T_{N-1}) < \dim_{\mathrm{aff}}(T_1, \ldots, T_N).$$

Proof.

$>$ is impossible, and $=$ would lead to existence of an equilibrium state supported on a proper subshift, which by the previous theorem is impossible as well. □

Strong irreducibility and quasimultiplicativity

Let $S \subseteq \mathrm{GL}(d, \mathbb{R})$ be the semigroup generated by $A_1, \ldots, A_N$.

S is **strongly irreducible** if it admits no nontrivial invariant finite union of subspaces.

Proposition

Suppose $\forall i \in \{1, \ldots, d-1\}$, the semigroup $\Lambda^i S$ is strongly irreducible. Then all singular value potentials are simultaneously quasimultiplicative, that is,

$$\exists c > 0 \ \exists \ell > 0 \ \forall B, C \in S \ \exists M \in S \text{ with } \mathrm{len}(M) \leq \ell \text{ s.t.}$$
$$\forall i \in \{1, \ldots, d-1\}, \quad \|\Lambda^i(BMC)\| \geq c \|\Lambda^i B\| \, \|\Lambda^i C\| \, .$$

A bit of algebraic geometry

Some important ingredients for the proof:

- The Zariski closure of a semigroup $S \subseteq \mathrm{GL}(d, \mathbb{R})$ is a group $G \subseteq \mathrm{GL}(d, \mathbb{R})$.
- If $G^\circ \subseteq G$ is the connected component of the identity, then:
  - G° is a group;
  - $[G : G^\circ] < \infty$.
  - G° is irreducible (as an algebraic variety).

Proposition (repeated)

Strong irr. on all ext. pow. ⇒ simultaneous quasi-mult.

Proof.

By contradiction. Mimicking a previous argument, we find $B_i, C_i \in \mathrm{End}(\Lambda^i \mathbb{R}^d)$ with $\|B_i\| = \|C_i\| = 1$ such that:

$$\forall M \in S \ \exists i \in \{1, \dots, d-1\} \text{ s.t. } \quad B_i(\Lambda^i M)C_i = 0.$$

Let $X_i := \{M \in \mathrm{GL}(d, \mathbb{R}) \, ; B_i(\Lambda^i M)C_i = 0\}$ (an algebraic set); then:

$$S \subseteq X_1 \cup \cdots \cup X_{d-1} \, .$$

Taking Zariski closure:

$$G \subseteq X_1 \cup \cdots \cup X_{d-1} \, .$$

Since G° is an irreducible component of G, it is contained in some X_i.

□

End of the proof.

We've just seen that $\exists i \in \{1, \ldots, d-1\}$ such that

$$G^\circ \subseteq X_i := \{M \in \mathrm{GL}(d, \mathbb{R})\ ;\ B_i(\Lambda^i M)C_i = 0\}$$

So $\Lambda^i G^\circ$ is reducible: the space

$$E := \mathrm{span}\,(\Lambda^i G^\circ)(\mathrm{Im}(C_i)) \subseteq \mathrm{Ker}(B_i)$$

is proper, nonzero, and $\Lambda^i G^\circ$-invariant.
Since $[G : G^\circ] < \infty$, the set $(\Lambda^i G)(E)$ is a $\Lambda^i G$-invariant finite union of proper nonzero subspaces of $\Lambda^i \mathbb{R}^d$.
This contradicts the strong irreducibility of $\Lambda^i S$. □

Rough ideas for the proof of our main theorem:

- Look action on $\bigoplus_i \Lambda^i \mathbb{R}^d$.
- Using block diagonalization, we essentially can assume each Λ^i action is irreducible.
- Consider the algebraic groups $G \supseteq G^\circ$.
- Morally, by passing to a finite cover, we can assume strong irreducibility.

Directions for further research

Question

Consider Hölder (fiber-bunched?) linear cocycles over a expanding or hyperbolic base dynamics. Is the number of ergodic equilibrium states of a singular value potential always finite?

In the case of a locally constant cocycle generated by a tuple of invertible matrices $(A_1, \ldots, A_N)$, it was important to consider the Zariski closure of the semigroup generated by the matrices, which is an algebraic subgroup $G \subseteq \mathrm{GL}(d, \mathbb{R})$.

Is there a similar tool for more general cocycles?

- Zimmer (80's) defined the **algebraic hull** of a measurable cocycle A as the smallest algebraic group G such that A is measurably conjugated to a G-valued cocycle.
- We can replace measurable class by Hölder class and obtain a **Hölder algebraic hull**.
- However, it seems difficult to "grab" this Hölder algebraic hull and so something useful with it. . .