Hypergeometric means and their completion

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What is a mean? An ad hoc definition

$\Omega =$ set (perhaps finite)

To "each" function $f : \Omega \to \mathbb{R}_+$, we want to associate a number $\mathfrak{m}(f)$ so that the following properties hold:

- **reflexivity**: $f = c$ (constant) $\Rightarrow \mathfrak{m}(f) = c$
- **monotonicity**: $f \leq g \Rightarrow \mathfrak{m}(f) \leq \mathfrak{m}(g)$
- **homogeneity**: $\forall \lambda \in \mathbb{R}_+, \ \mathfrak{m}(\lambda f) = \lambda \mathfrak{m}(f)$

Examples

- **Arithmetic mean**:

$$\mathfrak{am}(x_1, \ldots, x_n) := \frac{x_1 + \cdots + x_n}{n}$$

Functional version: if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then

$$\mathfrak{am}(f) = \mathbb{E}(f) := \int f \, d\mathbb{P} \qquad (a.k.a. \textbf{ expectation}).$$

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- **Geometric mean**:

$$\mathfrak{gm}(x_1, \ldots, x_n) := (x_1 \cdots x_n)^{\frac{1}{n}}$$

Functional version:

$$\mathfrak{gm}(f) := \exp \int \log f \, d\mathbb{P}$$

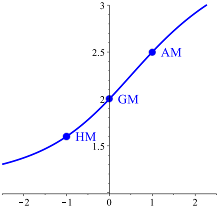
Hölder means (a.k.a. power means)

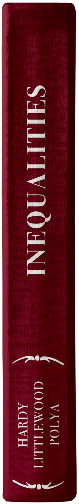
Definition

Given a positive function f on the probability space $(\Omega, \mathbb{P})$, the **Hölder mean** of f with parameter $p \in \mathbb{R}$ is:

$$\mathcal{M}_p(f) := \begin{cases} \left(\int f^p \, d\mathbb{P}\right)^{\frac{1}{p}} & \text{if } p \neq 0\,, \\ \exp \int \log f \, d\mathbb{P} & \text{if } p = 0\,. \end{cases}$$

Continuity and monotonicity wrt parameter:





Characterization of Hölder means

Given a mean $\mathfrak{m}$ and a homeomorphism ϕ (increasing or decreasing), consider a new (not necessarily homogeneous) mean

$$\widetilde{\mathfrak{m}}(f) := \phi^{-1}(\mathfrak{m}(\phi \circ f))$$

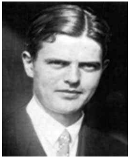
If $\mathfrak{m} = \mathfrak{am}$, then $\widetilde{\mathfrak{m}}$ is called a **quasi-arithmetic mean**. Example: $\widetilde{\mathfrak{m}} = \mathfrak{gm}$, $\phi = \log$.

Andrey Kolmogorov 1903–1987

Mitio Nagumo 1905–1993

Bruno de Finetti 1906–1985

Georg Aumann 1906–1980



Børge Jessen 1907–1993

Theorem (∼ 1930)

Hölder means are the only quasi-arithmetic homogeneous means.

A famous example of "nonlinear averaging"

Given numbers $b \geq a > 0$, define $[a_0, b_0] := [a, b]$ and recursively

$$[a_{n+1}, b_{n+1}] := [\mathfrak{gm}(a_n, b_n), \mathfrak{am}(a_n, b_n)] .$$

These intervals shrink superexponentially fast to a point c, called the **arithmetic geometric mean (AGM)** of a and b.

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Theorem (Lagrange 1785, Gauss 1800)

The AGM is related to an elliptic integral:

$$\frac{1}{\mathfrak{agm}(a,b)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}$$

Symmetric means

Using the elementary symmetric polynomials, we form the **symmetric means**:

$$\mathfrak{sym}_k(x_1, \ldots, x_n) := \left(\frac{\sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}}{\binom{n}{k}} \right)^{\frac{1}{k}}$$

Thus, $\mathfrak{sym}_1 \equiv \mathfrak{am}$ and $\mathfrak{sym}_n \equiv \mathfrak{gm}$.

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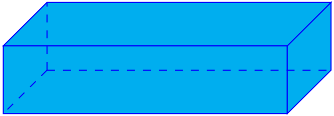
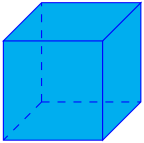


box with sides a, b, c

$\rightarrow$



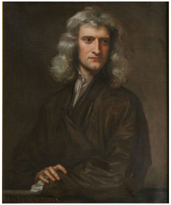
cube of side $\mathfrak{sym}_2(a, b, c) = \sqrt{\frac{ab+ac+bc}{3}}$

equal surface areas

Symmetric means (continued)

Monotonicity wrt parameter (Maclaurin, 1729):

$$\mathfrak{sym}_1 \geq \mathfrak{sym}_2 \geq \cdots \geq \mathfrak{sym}_{n-1} \geq \mathfrak{sym}_n$$



Isaac Newton
1642–1727

Colin Maclaurin
1698–1746

Limit laws for symmetric means?

Let $X_1, X_2, \ldots$ be a sequence of positive i.i.d. random variables (or, more generally, a stationary ergodic process). To avoid discussing integrability issues, let's assume uniform boundedness away from 0 and ∞.

By the Law of Large Numbers (or the Ergodic Theorem),

$$\frac{X_1 + \cdots + X_n}{n} \to \mathbb{E}(X) \quad \text{almost surely as } n \to \infty,$$

where X is a replica of the X_i's.

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Theorem

For any $k \geq 1$,

$$\mathfrak{sym}_k(X_1, \ldots, X_n) \to \mathbb{E}(X) \quad \text{a.s. when } n \to \infty.$$

Boring situation: No new limit. 😴

A more exciting limit law

Theorem (Hálasz–Székely 1976)

Let $k_1, k_2, \ldots$ be a sequence of integers such that $1 \le k_n \le n$ and k_n/n tends to some $c \in [0,1]$ as $n \to \infty$. Then, with probability 1,

$$\lim_{n\to\infty} \mathfrak{sym}_{k_n}(X_1, \ldots, X_n)$$

exists and equals a "computable" number $\mathfrak{HS}_{-c}(X)$ that depends only on the distribution of X and on the parameter c.

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Gábor Hálasz
1941–

Gábor J. Székely
1947–

Note: Székely was personally instigated by Kolmogorov to work on this (or a closely related) problem.

(Extended) Halász-Székely means

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

Let f be a positive function on Ω with $\|f^{\pm 1}\|_\infty < \infty$.

Definition

The **Halász-Székely mean** of f with parameter $\lambda \in \mathbb{R}$ is:

$$\mathfrak{HS}_\lambda(f) := \sup_{g>0} \left(\frac{\exp \int \log g \, d\mathbb{P}}{\int g \, d\mathbb{P}} \right)^{\frac{1}{\lambda}} \frac{\int fg \, d\mathbb{P}}{\int g \, d\mathbb{P}} \quad \text{if } \lambda > 0.$$

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If $\lambda = 0$, then $\mathfrak{HS}_0(f) := \int f \, d\mathbb{P}$.

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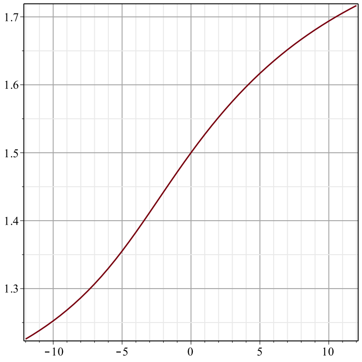
Note: $\mathfrak{HS}_\lambda(f)$ only depends on the measure $f_* \mathbb{P}$. It is a "nonlinear barycenter" of this measure.

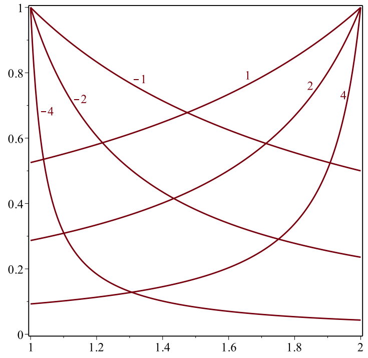
Special case: $\mathfrak{HS}_{-1}(f) = \exp \int \log f \, d\mathbb{P}$ (geometric mean).

Worked-out example

Plot of $\mathfrak{HS}_\lambda(f)$ as a function of λ if $\mathbb{P}$ = Lebesgue measure on $[1, 2]$ and $f(x) \equiv x$:



Functions $g = g_\lambda$ that attain the sup or inf (chosen so that $\max g = 1$):



How did I compute this?

Practical computation

Recall: $$\mathfrak{HS}_\lambda(f) := \begin{cases} \sup_{g>0} \left(\dfrac{\exp \mathbb{E}(\log g)}{\mathbb{E}(g)} \right)^{\frac{1}{\lambda}} \dfrac{\mathbb{E}(gf)}{\mathbb{E}(g)} & \text{if } \lambda > 0, \\ \mathbb{E}(f) & \text{if } \lambda = 0, \\ \inf \text{(same stuff)} & \text{if } \lambda < 0. \end{cases}$$

A positive function g that attains either the sup (if $\lambda > 0$) or the inf (if $\lambda < 0$) above is called an **equilibrium state**.

Proposition

Equilibrium states g are the essentially unique, and are characterized as the positive solutions of the eq.:

$$\frac{1}{g} + \frac{\lambda f}{\mathbb{E}(gf)} = \frac{1+\lambda}{\mathbb{E}(g)}$$

(which reduces to a scalar equation for $\xi := \mathbb{E}(fg)/\mathbb{E}(g)$...)

Limit theorems, revisited

Theorem (Halász–Székely 1976)

$$\frac{k_n}{n} \to c \in [0, 1] \quad \Rightarrow \quad \mathfrak{sym}_{k_n}(X_1, \ldots, X_n) \to \mathfrak{HS}_{-c}(X) \ a.s.$$

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A deterministic version:

Theorem (B.–Iommi–Ponce 2021)

Let X be a discrete random variable assuming distinct values $a_1, \dots, a_n$, each with probability $\frac{1}{n}$. If $1 \le k \le n$, then

$$\mathfrak{HS}_{-k/n}(X) \le \mathfrak{sym}_k(a_1, \dots, a_n) \le (9k)^{\frac{1}{2k}} \, \mathfrak{HS}_{-k/n}(X) \, .$$

Combining these inequalities with continuity properties of $\mathfrak{HS}$ (both as function of the parameter and the distribution), the previous theorem follows.

Mean rates of expansion of a linear operator

Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear. The **expansion rate** of A is the function

$$v \in S^{n-1} \mapsto \|Av\|$$

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$$\mathfrak{em}_p(A) := \left(\int_{S^{n-1}} \|Av\|^p \, d\sigma(v)\right)^{\frac{1}{p}} \quad \text{if } p \neq 0\,,$$

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Definition

The **ellipsoidal mean** with parameter p of numbers $a_1, \ldots, a_n \geq 0$ is $\mathfrak{em}_p(a_1, \ldots, a_n) = \mathfrak{em}_p(A)$, where $A = \operatorname{diag}(a_1, \ldots, a_n)$.

Some special cases

Recall:

$$\mathfrak{em}_p(a_1, \ldots, a_n) := \left(\int_{S^{n-1}} \|Av\|^p \, d\sigma(v) \right)^{\frac{1}{p}}, \quad A = \operatorname{diag}(a_1, \ldots, a_n).$$

Exercises:

$$\mathfrak{em}_2(A) = \sqrt{\tfrac{1}{n} \sum a_i^2} = \mathfrak{M}_2(a_1, \ldots, a_n)$$

$$\mathfrak{em}_{-n}(A) = \sqrt[n]{\prod a_i} = \mathfrak{M}_0(a_1, \ldots, a_n)$$

Are ellipsoidal means always Hölder? Definitely no!

Other special ellipsoidal means

- If $n = 2$, $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, then

$$
\begin{aligned}
\mathfrak{em}_{-1}(a, b) &= \left(\int_{S^1} \|Av\|^{-1} \, d\sigma(v) \right)^{-1} \\
&= \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \right)^{-1} \\
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- $\mathfrak{em}_1(a, b) = \frac{1}{2\pi} \cdot$ perimeter of the ellipse with semiaxes a, b (the quintessential elliptic integral).

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- Generalization: $\text{area}(A(S^{n-1})) = \mathfrak{em}_1(\Lambda^{n-1} A) \cdot \text{area}(S^{n-1})$.
- Also related to: mean width and electrostatic capacities of ellipsoids, spherical functions, Lyapunov exponents, Kullback-Leiber divergence.

Limit laws for ellipsoidal means?

Consider a sequence A_n of diagonal matrices of increasing dimensions $n \times n$

$$A_n = \operatorname{diag}(a_{n,1}, \ldots, a_{n,n}), \quad a_{n,i} > 0\,.$$

Assume that there exists a well-defined limit distribution of the diagonal entries:

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{a_{n,i}} \xrightarrow{\textit{weakly}} \mu \quad \text{as } n \to \infty\,.$$

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Theorem

If the numbers $a_{n,i}$ are bounded away from 0 and ∞, then, for any $p \in \mathbb{R}$,

$$\mathfrak{em}_p(A_n) \to \mathcal{M}_2(\mu) \quad \textit{as } n \to \infty .$$

Rivin 2004: More precise results (weaker assumption, estimates), but only for $p = 0$ or 1.

Proving that $\mathfrak{em}_p(A_n) \to \mathcal{M}_2(\mu)$

- Consider the expansion rate function

$$f_n \colon S^{n-1} \to \mathbb{R}_+ , \qquad f_n(v) := \|A_n v\| .$$

- As mentioned before, $\mathfrak{em}_2 \equiv \mathcal{M}_2$ (in particular, the theorem is trivial for $p = 2$). This means that f_n^2 has (arithmetic) mean equal to $\mathcal{M}_2(\mu_n)^2$.
- The function f_n^2 is Lipschitz (with bounds).

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This theorem isn't very exciting, since the limit is nothing new. 😴

An unexpected limit law for ellipsoidal means

As before, consider a sequence A_n of matrices of increasing dimensions $n \times n$, whose singular values $a_{n,i}$ are bounded away from 0 and ∞, and have a well-defined limit distribution.

Theorem

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What's the explanation for this miracle? Is there some relation between symmetric means and ellipsoidal means? Well, there is no direct relation (as far as I could determine). The clue is that these two means are members of a larger family...

Dirichlet distribution

Definition

The **Dirichlet distribution** with parameter $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_+$ is the probability measure $\theta_{\mathbf{b}}$ on the standard unit simplex

$$\Delta^{n-1} := \{(u_1, \ldots, u_n) \in \mathbb{R}^n_+ : u_1 + \cdots + u_n = 1\}$$

whose density wrt the area measure is proportional to the function

$$u_1^{b_1-1} \cdots u_n^{b_n-1} .$$

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Two special cases:

- If $\mathbf{b} = (1, \ldots, 1)$, then $\theta_{\mathbf{b}}$ = normalized area on Δ^{n-1}.

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$$u_1^{b_1 - 1} \cdots u_n^{b_n - 1}.$$

Two special cases:

- If $\mathbf{b} = (1, \ldots, 1)$, then $\theta_\mathbf{b}$ = normalized area on Δ^{n-1}.
- If $\mathbf{b} = (\frac{1}{2}, \ldots, \frac{1}{2})$, then $\theta_\mathbf{b}$ = the push-forward of normalized area on the sphere under the map

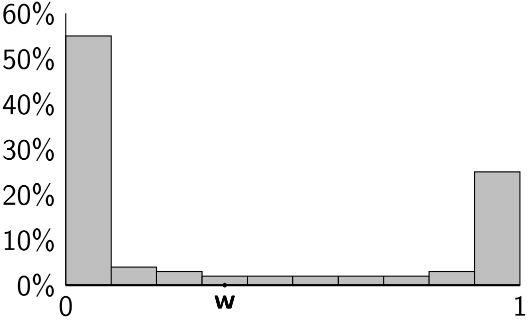
$$(x_1, \ldots, x_n) \in S^{n-1} \mapsto (x_1^2, \ldots, x_n^2) \in \Delta^{n-1}.$$

Limit cases

$$\mathbf{w} \in \Delta^{n-1} \quad \Rightarrow \quad \theta_{s\mathbf{w}} \text{ tends to } \begin{cases} \sum_i w_i \delta_{e_i} & \text{as } s \to 0^+ \\ \delta_{\mathbf{w}} & \text{as } s \to \infty \end{cases}$$

Visualization for $n = 2$, so Δ^{n-1} = an interval:

$s = 1/4$



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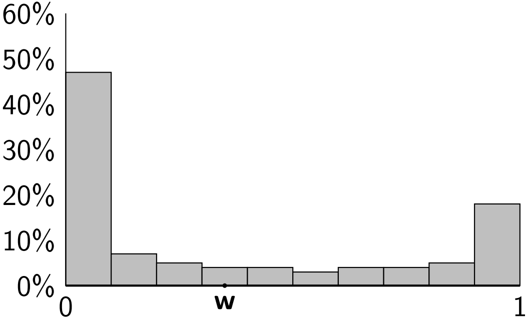

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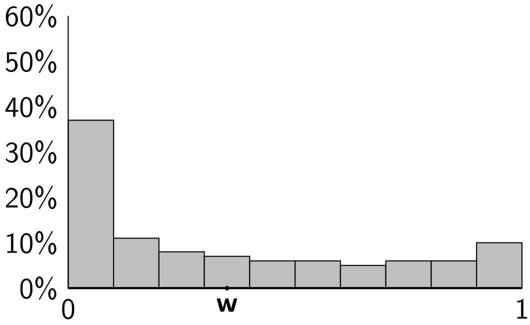

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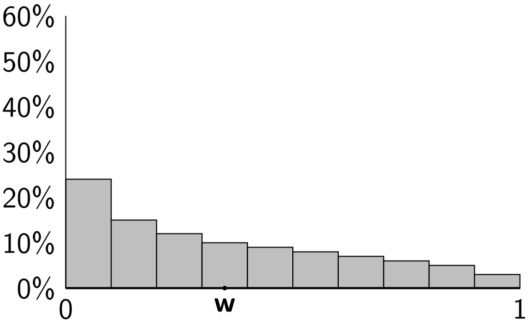

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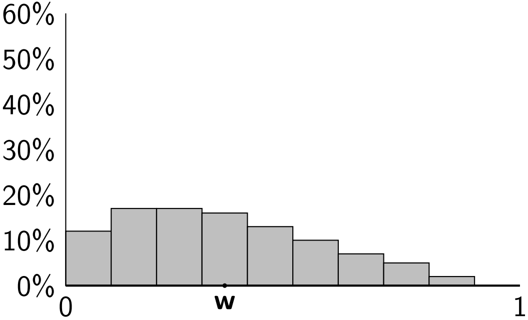
Visualization for $n = 2$, so $\Delta^{n-1} =$ an interval:

$s = \; 4$



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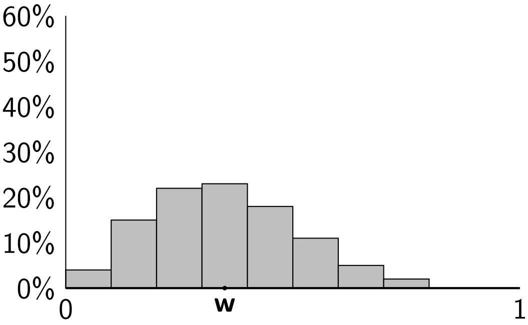

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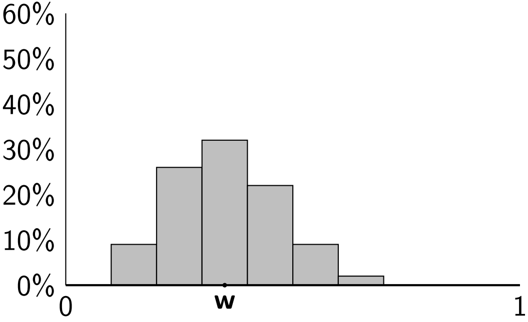
Visualization for $n = 2$, so Δ^{n-1} = an interval:

$s = 16$

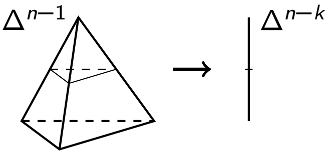


Properties (aggregation and neutrality)

Given $n > k > 1$, consider the map

$$\Delta^{n-1} \to \Delta^{n-k}$$
$$(u_1, \ldots, u_n) \mapsto (u_1 + \cdots + u_k, u_{k+1}, \ldots, u_n)$$

whose fibers are (scaled copies of) Δ^{k-1}.



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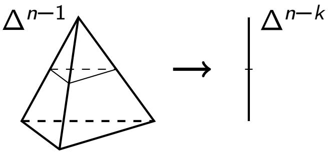

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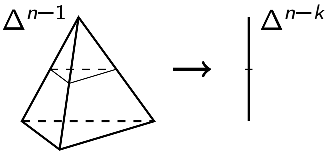

- **Aggregation property:** Dirichlet on Δ^{n-1} projects to Dirichlet on Δ^{n-k};
- **Neutrality property:** the conditional measures are Dirichlets.

Carlson's hypergeometric function R (1963)

Definition

Given $\mathbf{b} = (b_1, \ldots, b_n), \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, $t \in \mathbb{R}$, define:

$$R_t(\mathbf{b}; \mathbf{x}) := \int_{\Delta^{n-1}} \langle \mathbf{u}, \mathbf{x} \rangle^t \, d\theta_{\mathbf{b}}(\mathbf{u})$$

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A special case:

$$R_{\frac{p}{2}}\left(\tfrac{1}{2}, \cdots, \tfrac{1}{2}; a_1^2, \ldots, a_n^2\right) = \int_{S^{n-1}} \|Av\|^p \, d\sigma(v)$$

where $A = \operatorname{diag}(a_1, \ldots, a_n)$.

This follows from the change of variables $u_i = v_i^2$, which (as mentioned before) sends the area measure σ on S^{n-1} to the Dirichlet measure $\theta_{\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)}$ on Δ^{n-1}.

Properties of the function R

- homogeneity of degree t:

$$R_t(\mathbf{b}; \lambda \mathbf{x}) = \lambda^t R_t(\mathbf{b}; \mathbf{x})$$

- symmetry under permutations of indices;
- aggregation property:

$$x_1 = x_2 \;\Rightarrow\; R_t(\mathbf{b}, \mathbf{x}) = R_t(b_1 + b_2, b_3, \ldots, b_n; x_1, x_3, \ldots, x_n)$$

- and LOTS of others

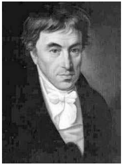
Classical hypergeometric function $F = {}_2F_1$ (Euler 1769)

$$x(1-x)\frac{d^2y}{dx^2} + [c-(a+b+1)x]\frac{dy}{dx} - aby = 0$$

$$F(a,b;c;x) = 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}\,du\,.$$


Euler
1707–1783


Pfaff
1765–1825


Gauss
1777–1855


Kummer
1810–1893


Riemann
1826–1866

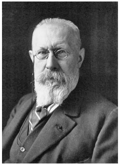
…

R versus F's

Carlson's R is related to Euler's $F = {}_2F_1$ if $n = 2$, to Appell's F_1 (1880) if $n = 3$, and to Lauricella's F_D (1893) for any n.

$$F(a, b; c; x) = R_{-a}(b, c - b; 1 - x, 1)$$
$$R_t(b_1, b_2; x_1, x_2) = x_2^t F(-t, b_1; b_1 + b_2; 1 - \frac{x_1}{x_2})$$

"The symmetry of R entails the cost of an extra variable resulting from homogeneous coordinates. To use $ax^2 + bxy + cy^2$ instead of $ax^2 + bx + c$ would be analogous."



Paul Émile Appell
1855–1930

Giuseppe Lauricella
1867–1913

Bille Chandler Carlson
1924–2013

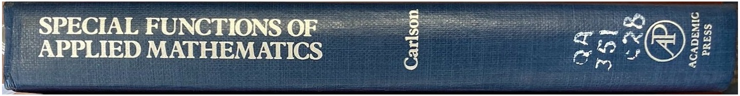
Advantages of symmetry

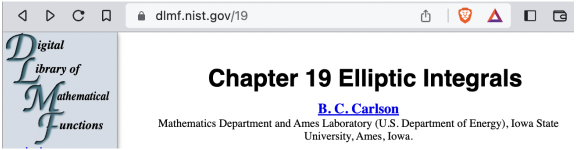
One example (among many): Pfaff's reflection law

$$\frac{1}{(1-z)^a} F\left(a, b; c; -\frac{z}{1-z}\right) = F(a, c-b; c; z) \qquad (1)$$

becomes

$$R_t(b_1, b_2; z_1, z_2) = R_t(b_2, b_1; z_2, z_1)\,. \qquad (2)$$





Carlson's hypergeometric means (1964)

Given an tuple of positive numbers $\mathbf{x}$, a weight vector $\mathbf{w} \in \Delta^{n-1}$, and parameters $t \in \mathbb{R} \setminus \{0\}$ (the **exponent**), $s > 0$ (the **concentration**), the **hypergeometric mean** is

$$\mathfrak{hygm}_{t,s}(\mathbf{x}, \mathbf{w}) := [R_t(s\mathbf{w}; \mathbf{x})]^{\frac{1}{t}} = \left(\int_{\Delta^{n-1}} \langle \mathbf{u}, \mathbf{x} \rangle^t \, d\theta_{\mathbf{b}}(\mathbf{u}) \right)^{\frac{1}{t}}$$

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Note that this is a t-Hölder wrt the Dirichlet measure $\theta_{s\mathbf{w}}$. So it makes sense to extend it to $t = 0$ as the corresponding geometric mean:

$$\mathfrak{hygm}_{0,s}(\mathbf{x}, \mathbf{w}) := \exp \int_{\Delta^{n-1}} \log \langle \mathbf{u}, \mathbf{z} \rangle \, d\theta_{\mathbf{b}}(\mathbf{u})$$

(related to another hypergeometric function $L_0 := \left.\frac{\partial R_t}{\partial t}\right|_{t=0}$).

Hypergeometric means extend ellipsoidal means

$$\mathfrak{em}_p(a_1,\ldots,a_n) = \sqrt{\mathfrak{hygm}_{\frac{p}{2},\frac{n}{2}}\left(a_1^2,\ldots,a_n^2;\frac{1}{n},\cdots,\frac{1}{n}\right)}.$$

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The hypergeometric mean is more flexible: s doesn't need to be a half-integer, and it allows for weights.

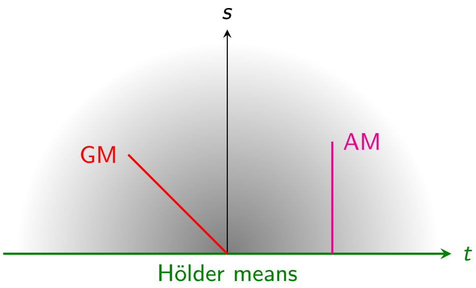
Note that weights work as they should (by the aggregation property of R).

Special cases of the hypergeometric mean

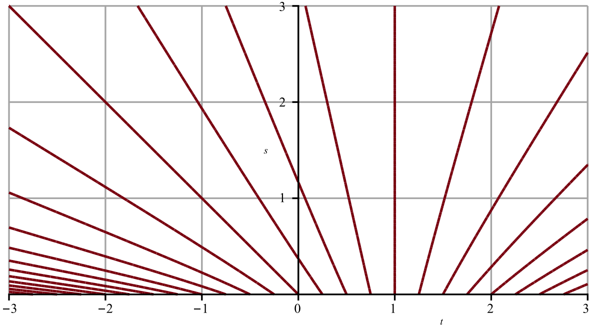
$\mathfrak{hygm}_{t,s}$ can be defined on the closed half-plane $s \geq 0$.

Special cases:

- Concentration $s = 0 \Rightarrow$ Hölder mean with exponent t
- Exponent $t = 1 \Rightarrow$ arithmetic mean
- $t = -s \Rightarrow$ geometric mean
- and many other particular means...



Examples: level sets



(Levels correspond to Hölder means with spacing $\frac{1}{4}$)

The levels sets are **not** straight lines, except for $s = -t$ (GM), $t = 1$ (AM) (and $t = \frac{1}{2} - \frac{s}{2}$ if $n = 2$)).

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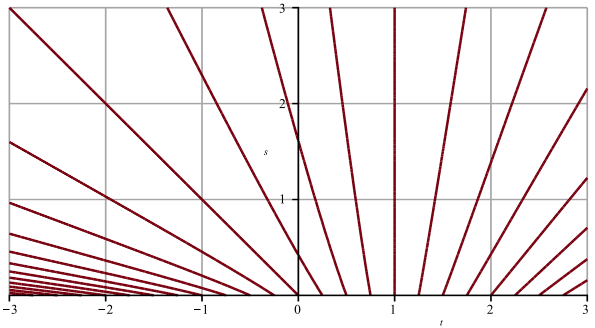

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Boring limit theorems

Theorem (Carlson 1964)

For any fixed $t \in \mathbb{R}$,

$$\lim_{s \to +\infty} \mathfrak{hygm}_{t,s}(\mathbf{x}, \mathbf{w}) = \mathfrak{am}(\mathbf{x}, \mathbf{w}) \,.$$

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Theorem (Brenner–Carlson 1987)

Suppose μ_n *is a sequence of discrete probability measures supported on a common compact subinterval of* $\mathbb{R}_+$, *and converging weakly to some probability* μ. *Let* $t \in \mathbb{R}$ *be fixed and* $s = s(n) \to +\infty$. *Then,*

$$\lim_{n \to \infty} \mathfrak{hygm}_{t,s(n)}(\mu_n) = \mathfrak{am}(\mu)\,.$$

No new limits 😴

Exciting limit theorem

Theorem

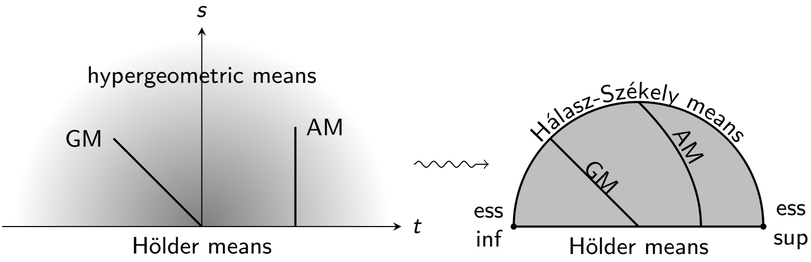
Suppose μ_n is a sequence of discrete probability measures supported on a common compact subinterval of $\mathbb{R}_+$, and converging weakly to some probability μ. If $t = t(n)$ and $s = s(n) \to +\infty$ are such that $t/s \to \lambda \in \mathbb{R}$, then

$$\lim_{n\to+\infty} \mathfrak{hygm}_{t(n),s(n)}(\mu_n) = \mathfrak{HS}_\lambda(\mu).$$

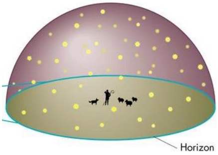
Disclaimer: $\exists$ related results on approximation of hypergeometric means using saddle point method: Jiang–Kadane–Dickey 1991, Butler–Wood 2015

A compactification of the space of parameters

Projective compactification of the space of parameters (s, t): from a half-plane to a half-disk.



This is like a "celestial sphere":



The theory is still incomplete...

Note that the Hálasz-Székely means are "functional" means (or "barycenters").

What about the hypergeometric means – do they admit a functional version?

The answer is yes!

Short explanation: Any homogeneous mean coherently defined for weighted finite lists of arbitrary lengths can be extended to a functional mean (i.e. barycenter).

To find this extension concretely, the first step is to extend the Dirichlet measures...

Ferguson-Dirichlet process

The simplex Δ^{n-1} is the space of probability measures on the finite set $F = \{1, \ldots, n\}$. Therefore, each Dirichlet $\theta_{\mathbf{b}}$ is a probability distribution on set of probabilities on F (think "a bag of loaded dice").

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Theorem (Freedman 1963, Fabius 1964, Ferguson 1973)

Let β be a positive finite measure on $(\Omega, \mathcal{F})$. Then there exist a ***random probability measure*** *μ on Ω such that, for any finite measurable partition $A_1 \sqcup \cdots \sqcup A_n = X$, the distribution of the random vector $(\mu(A_1), \ldots, \mu(A_n)) \in \Delta^{n-1}$ is Dirichlet with parameter $(\beta(A_1), \ldots, \beta(A_n))$.*

This result produces a **probability measure on the space of probability measures on** Ω, called **Ferguson-Dirichlet measure** and denoted Θ_β.

Key: the aggregation property of the (finite-dimensional) Dirichlet.

Infinite-dimensional hypergeometric stuff

β = a finite measure on $\mathbb{R}_+$

$$R_t(\beta) := \int_{\mathcal{P}(\mathbb{R}_+)} \left(\int_{\mathbb{R}_+} x \, d\mu(x) \right)^t d\Theta_\beta(\mu)$$

If β is a discrete measure $\sum b_i \delta_{x_i}$, we get the previous $R_t(\mathbf{b}, \mathbf{x})$.

Disclaimer: Many people considered averages wrt Ferguson–Dirichlet. It was already known that hypergeometric functions have an infinite dimensional generalization: Lijoi–Regazzini 2004

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$\mu =$ a probability measure on $\mathbb{R}_+$

$$\mathfrak{hygm}_{t,s}(\mu) := [R_t(s\mu)]^{\frac{1}{t}}$$

Completion of hypergeometric means

Theorem (Main theorem)

The hypergeometric means can be extended to a ***continuous*** *map*

$$\mathfrak{hygm} : H \times \mathcal{M}_I \to \mathbb{R}_+ \ ,$$

where:

- *H is the closed half-disk (projective compactification of the half-plane);*
- *$\mathcal{M}_I$ is the space of probability measures supported on an interval $I = [a, b] \subset \mathbb{R}_+$*

What about symmetric means?

Fact: $\mathfrak{sym}_k(x_1, \ldots, x_n) = \mathfrak{hygm}_{k,-n}\left(x_1, \ldots, x_n; \frac{1}{n}, \cdots, \frac{1}{n}\right)$

Despite $s < 0$, the RHS makes sense, because:

- R_k is a polynomial in the x_i variables, and
- $R_k(-1, \ldots, -1; x_1, \ldots, x_n) > 0$ if all $x_i > 0$.

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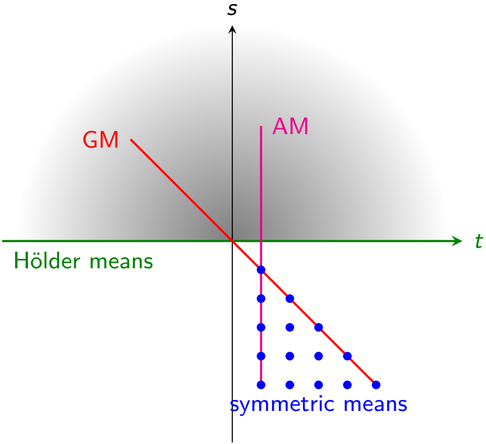
So the limit formula also works in this case ("below the horizon").



Matrix arguments?

B.–Iommi–Ponce 2016 obtained the $\mathfrak{sym} \to \mathfrak{HS}$ law as a particular case of a "**law of large permanents**":

$$\left(\frac{1}{n!}\,\mathrm{per}(A_n)\right)^{\frac{1}{n}} \to \text{"scaling mean"}$$

(later improved by Balogh–Nguyen 2017) – that's a topic for another talk.

Question

Is this all of this part of something even bigger?

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Is this all of this part of something even bigger?

In an effort to symmetrize other classical hypergeometric functions, Carlson (1971) introduced a hypergeometric function $\mathcal{R}$ of matrix argument. **It turns out that the permanent is a particular case of the $\mathcal{R}$ function.**