STRUCTURES INDUCED BY THE SYMMETRIC DIFFERENCE METRIC

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This note was originally written as an appendix to the first version of the paper [1].

1. Intrinsic metrics

Given any metric space (M, d), we can define the (possibly infinite) *length* of a continuous curve $f\colon [0,1] \to M$ as

$$\operatorname{len}_d(f) \coloneqq \sup \sum_{j=0}^{n-1} d(f(t_j), f(t_{j+1}))\,,$$

where the supremum is taken over all choices of points $0 = t_0 < t_1 < \cdots < t_n = 1$. If M is path-connected then we define, for every x, $y \in M$

$$\bar{d}(x,y) \coloneqq \inf_f \operatorname{len}_d(f)\,,$$

where f runs over all the continuous curves from x to y.

Note that $\bar{d} \geqslant d$ and $\bar{d}$ satisfies the usual axioms of a metric, except that it may be infinite. (Actually, it may be identically ∞: consider for example the Koch curve.) If $\bar{d}$ is everywhere finite and it induces the same topology as d, then we call it the *intrinsic metric* induced by d. A curve f that attains the infimum in the formula above (if it exists) is a called a (parameterized) *geodesic segment* joining x to y. See [4, Chapter 2] for much more information on this subject.

2. Geometry of the space of ellipses

Let $\mathcal{E}$ be the set of ellipses in $\mathbb{R}^2$ with center at the origin and area π. We endow this set with the *symmetric difference metric*:

$$d(E, E') \coloneqq \operatorname{area}(E \vartriangle E')\,,$$

This is indeed a metric: the triangle inequality follows from the set inclusion $E \vartriangle E'' \subseteq (E \vartriangle E') \cup (E' \vartriangle E'')$. Of course, we are only adapting the Fréchet–Nikodym metric from Measure Theory (see e.g. [2, p. 53]) to the context of ellipses.

Proposition 1. *$(\mathcal{E}, d)$ induces an intrinsic metric, which is given by the formula:*

$$\bar{d} = 4 \log \tan \left(\frac{d}{8} + \frac{\pi}{4} \right) .$$

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We call $\bar{d}$ the *footprint metric* on $\mathcal{E}$; the name comes from a geometric description of the length of curves that we will present later (see Proposition 4).

In order to prove Proposition 1, we will actually show that $(\mathcal{E}, \bar{d})$ is a well-known metric space in disguise.

Let $\mathbb{H} := \{z \in \mathbb{C} \ ; \ \operatorname{Im} z > 0\}$ be the upper-half plane, endowed with the Riemannian metric obtained by multiplying the euclidian inner product by $(\operatorname{Im} z)^{-2}$. Let $d_{\mathbb{H}}(\cdot, \cdot)$ be the corresponding distance function. This space has constant curvature -1, and its geodesics are either euclidian half-lines or euclidian half-circles orthogonal to the real axis. Moreover, the group $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathbb{H}$ by isometries via Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az + b}{cz + d} \, .$$

The action is transitive, and the stabilizer of $i = \sqrt{-1}$ is $\mathrm{SO}(2)$, the group of rotations. Since the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathcal{E}$ is also transitive, and the stabilizer of the unit disk is also $\mathrm{SO}(2)$, there exists an unique bijection

$$\psi \colon \mathbb{H} \to \mathcal{E}$$

that sends i to the unit disk and that commutes with the group action (i.e., $\psi(LE) = L\psi(E)$).

Proposition 2. *ψ defines an isometry between $(\mathbb{H}, 2d_{\mathbb{H}})$ and $(\mathcal{E}, \bar{d})$.*

The first step of the proof is the following computation:

Lemma 3. *Let $E_0 \in \mathcal{E}$ be the unit disk, and for each $t \in \mathbb{R}$ define $E_t \in \mathcal{E}$ by:*

$$E_t := \big\{ (x, y) \in \mathbb{R}^2 \ ; \ e^t x^2 + e^{-t} y^2 \leqslant 1 \big\}.$$

Then:

$$d(E_0, E_t) = -2\pi + 8 \arctan e^{|t|/2} \, .$$

Proof. Initially, consider an arbitrary pair of distinct numbers t, $s \in \mathbb{R}$. Then $\partial E_t \cap \partial E_s$ consists on four points $\pm p$, $\pm q$. Consider the following four lines: the two coordinate axes, the line containing $\pm p$, and the line containing $\pm q$. These four lines split the region $E_t \cap E_s$ into eight "slices". We claim that all slices have the same area. In the case that $t + s = 0$, this claim is clearly true by symmetry. Applying the diagonal subgroup of $\mathrm{SL}(2, \mathbb{R})$, we conclude that the claim always holds.

Now consider $s = 0$. Then four of the slices are sectors of the unit disk E_0 with central angles $\theta := \arctan e^{-|t|/2} = \frac{\pi}{2} - \arctan e^{|t|/2}$, and therefore all eight slices have area $\theta/2$. So:

$$d(E_0, E_t) = 2\pi - 2 \operatorname{area}(E_0 \cap E_t) = 2\pi - 8\theta \, ,$$

yielding the announced formula. □

Proof of Propositions 1 and 2. Pulling back the symmetric difference metric d by the map ψ we obtain a new metric on $\mathbb{H}$:

$$(\psi^* d)(z_1, z_2) := d\big(\psi(z_1), \psi(z_2)\big) \, .$$

For any pair of points z_1, $z_2 \in \mathbb{H}$ there exists $L \in \mathrm{SL}(d, \mathbb{R})$ such that $L(z_1) = i$ and $L(z_2) = e^{-t} i$ for some $t \in \mathbb{R}$. Note that $\psi(e^{-t} i) = E_t$ and that $d_{\mathbb{H}}(e^{-t} i, i) = |t|$. So it follows from Lemma 3 that

$$(\psi^* d)(z_1, z_2) = -2\pi + 8 \arctan e^{d_{\mathbb{H}}(z_1, z_2)/2} \, .$$

In particular,

$$(\psi^* d)(z_1, z_2) = 2d_{\mathbb{H}}(z_1, z_2) + o\big(d_{\mathbb{H}}(z_1, z_2)\big) .$$

This property clearly implies that $\psi^* d$ induces the intrinsic metric $\overline{\psi^* d} = 2d_{\mathbb{H}}$. Propositions 1 and 2 follow. ☐

3. Geometric interpretation of the length with respect to the symmetric difference metric

Proposition 4. *Suppose that* $F\colon [0,1] \to \mathcal{E}$ *is a continuous curve. Then the length of* F *is given by the area integral*

$$\operatorname{len}_d(F) = \iint_{\mathbb{R}^2} N(x,y)\, \mathrm{d}x\, \mathrm{d}y\,,$$

where $N(x,y) \coloneqq \operatorname{card}\{t \in [0,1] \,;\, (x,y) \in \partial F(t)\}$.

In informal terms, $\operatorname{len}_d(F)$ measures the area of $\bigcup_{t\in[0,1]} \partial F(t)$ with multiplicities; thus we call it the *footprint length* of the path F. To illustrate, Fig. 1 shows the case where F is a parametrized segment $[E_{-c}, E_c]$.

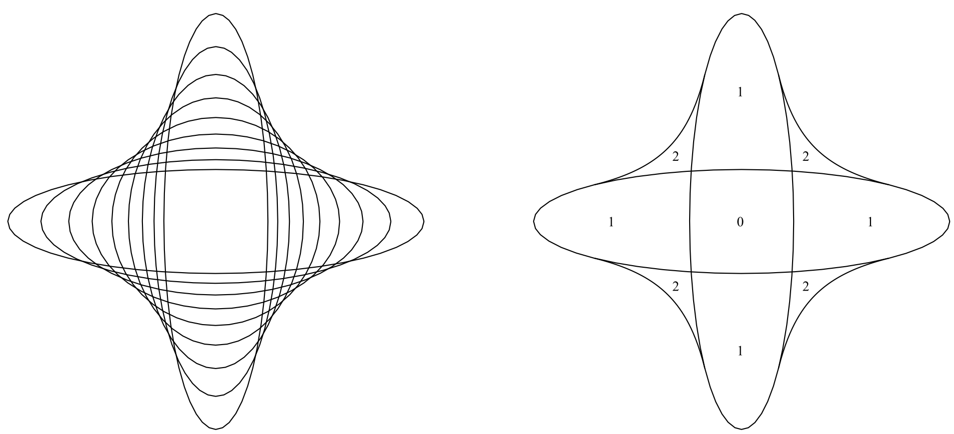

Figure 1. A path F and the associated function N.

The proof of Proposition 4 is omitted.

4. Other spaces of sets

The footprint characterization of length given by Proposition 4 also makes sense in bigger spaces of sets.

Here is an interesting example: Let $\mathcal{K}$ be the set of all subsets $K \subset \mathbb{R}^2$ that are compact, convex, centrally symmetric (i.e. $K = -K$), and have area π. Endow $\mathcal{K}$ with the symmetric difference metric d, defined exactly as before. Then the statement of Proposition 4 holds in this setting.

A caveat: Unlike $(\mathcal{E}, d)$, geodesic segments are not unique in the space $(\mathcal{K}, d)$: for example if t is sufficiently small then there exist many curves in $\mathcal{K}$ from E_0 to E_t with footprint length equal to $d(E_0, E_t) = \operatorname{area}(E_0 \vartriangle E_t)$.

5. Higher dimensional ellipsoids

We briefly comment on how what was done above extends (or not) to higher dimension, omitting the proofs.

Let $\mathcal{E}_n$ be space of ellipsoids in $\mathbb{R}^n$ with center at the origin and the same volume as the unit ball, endowed with the symmetric difference metric d_n. There is an associated intrinsic metric $\bar{d}_n$; moreover geodesic segments always exist and are unique modulo reparametrization. There is a bijection ψ_n between the symmetric space $P_n \coloneqq \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ and the space $\mathcal{E}_n$ that sends the basepoint to the unit ball and that commutes with the action of $\mathrm{SL}(d, \mathbb{R})$. Consider P_n endowed with its unique $\mathrm{SL}(d, \mathbb{R})$-invariant Riemannian metric (which has nonpositive curvature: see e.g. [3, Chapter II.10]). Then ψ_n is *not* an isometry, but it is affine: it preserves geodesic segments. The pull-back of $\bar{d}_n$ to P_n is the distance function induced by a $\mathrm{SL}(d, \mathbb{R})$-invariant Finsler non-Riemannian structure. The lengths of curves have a footprint characterization similar to Proposition 4.

References

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