Flexibility of Lyapunov exponents

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SETTING AND QUESTIONS

A. Katok's flexibility program

Fix:

- a class of smooth dynamical systems (action of $\mathbb{Z}_+$ or $\mathbb{Z}$ or $\mathbb{R}$);
- one or more dynamically invariant quantities (like entropies or Lyapunov exponents with respect to a relevant measure).

Flexibility paradigm:

> *There should be no restrictions on the dynamical quantities apart from a few "obvious" ones.*

☞Alena Erchenko's talk yesterday.

Setting for today: conservative diffeos; Lyapunov exponents

- M = compact connected manifold of dimension $d \geq 2$.
- m = normalized volume measure on M.

If $f : M \to M$ is a conservative (i.e., m-preserving) ergodic diffeomorphism, the *Lyapunov exponents* are:

$$\boxed{\lambda_i(f) := \lim_{n \to +\infty} \frac{1}{n} \log\big(i\text{-th singular value of } Df^n(x)\big)}$$

(for m-a.e. $x \in M$).

Note: $\boxed{\lambda_1(f) \geq \cdots \geq \lambda_d(f)}$ and $\boxed{\sum_{i=1}^{d} \lambda_i(f) = 0}$.

- *Lyapunov spectrum* $\vec{\lambda}(f) = \big(\lambda_1(f), \dots, \lambda_d(f)\big)$.
- The Lyapunov spectrum is called *simple* if these numbers are all different.

Flexibility conjectures

Problem

Which Lyapunov spectra $\vec{\lambda}(f) = \big(\lambda_1(f), \ldots, \lambda_d(f)\big)$ *may appear for* C^∞ *ergodic diffeomorphisms* f*?*

Apart from the obvious restrictions that the λ_i's should be ordered and their sum should be zero, no other conditions are known.

Conjecture (Weak flexibility – general)

Fix (M, m)*. Given* $\xi_1 \geq \cdots \geq \xi_d$ *with* $\sum_i \xi_i = 0$*, then there exists ergodic* $f \in \mathrm{Diff}^\infty_m(M)$ *such that* $\boxed{\vec{\lambda}(f) = (\xi_1, \ldots, \xi_d)}$.

Existence of ergodic smooth diffeomorphisms

- All exponents zero: Anosov–Katok (early 70's)
- No exponents zero: Katok (1979) for $d = 2$; Dolgopyat–Pesin (2002)

Flexibility conjectures

Even more ambitious: fix homotopy class.

Conjecture (Strong flexibility – general)

Fix (M, m). Fix a connected component $\mathcal{C} \subseteq \mathrm{Diff}^\infty_m(M)$. Given $\xi_1 \geq \cdots \geq \xi_d$ with $\sum_i \xi_i = 0$, then there exists $\boxed{\text{ergodic } f \in \mathcal{C}}$ such that $\boxed{\vec{\lambda}(f) = (\xi_1, \ldots, \xi_d)}$.

Terminology:

- "Strong" means prescribed homotopy class.
- "Weak" means we don't care about homotopy class

Flexibility conjectures for Anosov diffeos on $\mathbb{T}^d$

Let's work on the more manageable class of *conservative Anosov smooth diffeomorphisms* (which are automatically ergodic).

Conjecture (Weak flexibility – Anosov)

Given $\xi_1 \geq \cdots \geq \xi_d$ with $\sum_i \xi_i = 0$ and each $\xi_i \neq 0$, then there exists an Anosov $f \in \mathrm{Diff}^\infty_m(\mathbb{T}^d)$ such that $\boxed{\vec{\lambda}(f) = (\xi_1, \ldots, \xi_d)}$.

As a corollary of our main result, we prove this conjecture in the case of simple spectrum: $\xi_1 > \cdots > \xi_d$.

For Anosov, there is a new "obvious" restriction (given the homotopy class)

Given a conservative smooth Anosov $f\colon \mathbb{T}^d \to \mathbb{T}^d$, take $L = \pi_1(f) \in \mathrm{GL}(d,\mathbb{Z})$; then f is homotopic (and topologically conjugate) to the automorphism $F_L\colon \mathbb{T}^d \to \mathbb{T}^d$. Let u be the unstable index ($\dim E^u$) of either f or F_L. Then:

$$\boxed{\sum_{i=1}^{u} \lambda_i(f) \leq \sum_{i=1}^{u} \lambda_i(L)}$$

"entropy condition"

Proof:

$$\begin{aligned} \textstyle\sum_{i=1}^{u} \lambda_i(f) &= h_m(f) && \text{(Pesin's formula)} \\ &\leq h_{\mathrm{top}}(f) && \text{(variational principle)} \\ &= h_{\mathrm{top}}(F_L) && \text{(topological conjugacy)} \\ &= \textstyle\sum_{i=1}^{u} \lambda_i(L) \end{aligned}$$

Strong flexibility for Anosov?

Are there other restrictions?

Problem (Strong flexibility – Anosov)

Let $L \in \mathrm{GL}(d, \mathbb{Z})$ be hyperbolic matrix with unstable index u. Given $\xi_1 \geq \cdots \geq \xi_u > 0 > \xi_{u+1} \geq \cdots \geq \xi_d$ such that

$$\sum_{i=1}^{d} \xi_i = 0 \quad \textit{and} \quad \sum_{i=1}^{u} \xi_i \leq \sum_{i=1}^{u} \lambda_i(L),$$

does there exist a conservative Anosov diffeomorphism f homotopic to F_L such that $\vec{\lambda}(f) = (\xi_1, \ldots, \xi_d)$?

☞More about this problem in a couple of minutes.

A RESULT FOR $\mathbb{T}^3$

Dominated splittings

A *simple dominated splitting (SDS)* for $f \in \mathrm{Diff}^1_m(M)$ is a Df-inv. splitting

$$TM = E_1 \oplus \cdots \oplus E_d \quad \text{with each } \dim E_i = 1$$

such that $\exists n_0 > 0$ s.t. $\forall x \in M$ and all unit vectors $v_1 \in E_1(x), \dots, v_d \in E_d(x)$,

$$\|Df^{n_0}(v_1)\| > \cdots > \|Df^{n_0}(v_d)\|.$$

Then Lyapunov exponents are given by integrals:

$$\lambda_i(f) = \int \log \|Df|_{E_i}\| \, dm$$

and the spectrum is simple: $\lambda_1(f) > \cdots > \lambda_d(f)$.

Flexibility for Anosov with SDS on $\mathbb{T}^3$

Theorem (B-K-RH)

Fix $L \in \mathrm{GL}(3,\mathbb{Z})$ hyperbolic matrix with simple spectrum. Suppose $\xi_1 > \xi_2 > \xi_3$ have the same signs as $\lambda_1(L) > \lambda_2(L) > \lambda_3(L)$,

$$\begin{aligned} \xi_1 &\leq \lambda_1(L), \\ \xi_1 + \xi_2 &\leq \lambda_1(L) + \lambda_2(L), \quad \text{and} \\ \xi_1 + \xi_2 + \xi_3 &= 0\,. \end{aligned}$$

Then there exists a Anosov $f \in \mathrm{Diff}^\infty_m(T)$ with SDS homotopic to F_L such that $\vec{\lambda}(f) = (\xi_1, \xi_2, \xi_3)$.

*Furthermore, **the converse holds**.*

Note that there is an **extra** not-so-obvious inequality (related to SDS).

Proof of the "converse" (inequalities are necessary)

- Taking inverses if necessary, assume $\lambda_2(L) > 0$, i.e., $\dim E^{\mathrm{u}} = 2$.
- Then $\lambda_1(f) + \lambda_2(f) \le \lambda_1(L) + \lambda_2(L)$ is the "entropy condition".
- By contradiction, suppose that $\lambda_1(f) > \lambda_1(L)$.
- For a.e. x, and $n \gg 1$ the curve $\Gamma = f^n(W^{uu}_{\mathrm{loc}}(x))$ has length $\gtrsim e^{\lambda_1(f)n}$.
- The distance between the endpoints of the lifted curve $\tilde{\Gamma} \subset \mathbb{R}^3$ is $\sim e^{\lambda_1(L)n}$ (much smaller).
- This contradicts Brin–Burago–Ivanov'09 ($\widetilde{W}^{uu}$ leaves are quasi-isometric).

An exotic Anosov diffeomorphism?

Here is a more modest version of the Problem “Strong Flexibility – Anosov”:

Problem

Is there a C^∞ conservative Anosov diffeo of $\mathbb{T}^3$ with $\dim E^u = 2$ and $\lambda_1(f) > \lambda_1(L)$ (where $L \in \mathrm{GL}(3, \mathbb{Z})$ is the homotopy type)?

☞ f cannot have a simple dominated splitting, so it cannot be a C^1-perturbation of its linear part.
☞ The Pesin 1-dim manifolds $W^{uu}(x)$ should be very twisted inside the 2-dim leaves $W^u(x)$.

Idea: Try f = L^1-perturbation of another (well-chosen) linear Anosov. . .

MAIN RESULT

The majorization partial order

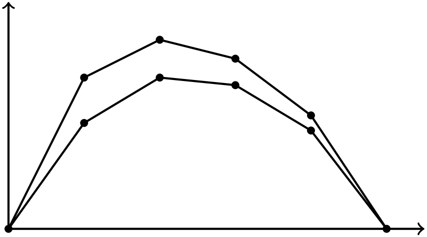
Let $\vec{\xi} = (\xi_1, \ldots, \xi_d)$ be an ordered vector ($\xi_i \geq \xi_{i+1}$) with $\xi_1 + \cdots + \xi_d = 0$.

Define a partial order on the set of such vectors:

$$\vec{\xi} \preccurlyeq \vec{\eta} \quad \Leftrightarrow \quad \xi_1 + \cdots + \xi_i \leq \eta_1 + \cdots + \eta_i \quad \forall i \in \{1, \ldots, d-1\}.$$

We say $\vec{\xi}$ *is majorized by* $\vec{\eta}$.
If the inequalities are strict: $\vec{\xi} \prec \vec{\eta}$ ($\vec{\xi}$ *is strictly majorized by* $\vec{\eta}$.)

Two concave graphs, one above the other:



Our main result

Let M be a compact manifold. Let $\mathcal{AS} \subset \mathrm{Diff}^\infty_m(M)$ be be formed by Anosov diffeomorphisms with SDS (simple dominated splitting).

Theorem (B,K,RH)

Let $f \in \mathcal{AS}$; let $u = \dim E^{\mathrm{u}}$.
Let $\vec{\xi} = (\xi_1, \dots, \xi_d)$ be such that:

$$\xi_1 > \dots > \xi_u > 0 > \xi_{u+1} > \dots > \xi_d$$

$$\xi_1 + \dots + \xi_d = 0,$$

$$\vec{\xi} \prec \vec{\lambda}(f) \text{ (strict majorization)}$$

Then there exists a continuous path $(f_t)_{t\in[0,1]}$ in $\mathcal{AS}$ starting from $f_0 = f$ such that $\vec{\lambda}(f_1) = \vec{\xi}$.

Keywords of the proof

The proof is essentially a optimized and global version of **Baraviera–Bonatti** perturbation method, which needs:

- special adapted metrics (a la Gourmelon) with a new "L^1-property";
- careful linear algebra (in order to mix several exponents simultaneously);
- tower methods (Rokhlin $+$ Vitali).

☞ More details later.

Corollary: Weak flexibility on $\mathbb{T}^d$

Corollary

For all nonzero numbers $\xi_1 > \cdots > \xi_d$ whose sum is 0, there exists a C^∞ conservative Anosov diffeo $f\colon \mathbb{T}^d \to \mathbb{T}^d$ with SDS such that $\vec{\lambda}(f) = \vec{\xi} := (\xi_1, \ldots, \xi_d)$.

Proof.

Given $\vec{\xi}$, we take a linear Anosov $L \in \mathrm{SL}(d, \mathbb{Z})$ with the same unstable index, and "large" enough so that:

$$\vec{\lambda}(L) \succ \vec{\xi}.$$

Then we apply the Main Theorem. □

PROOF

Review of Baraviera–Bonatti

As the proof of our main result relies on the Baraviera–Bonatti strategy, let us recall (a particular case of) their result:

Theorem (Baraviera–Bonatti, 2003)

Let f be a stably ergodic C^∞ conservative diffeomorphism with a simple dominated splitting. Then, for each $i \in \{1, \ldots, d\}$, there exists a C^∞ conservative diffeomorphism $\tilde{f}$ arbitrarily C^1-close to f such that $\lambda_i(\tilde{f}) \neq \lambda_i(f)$.

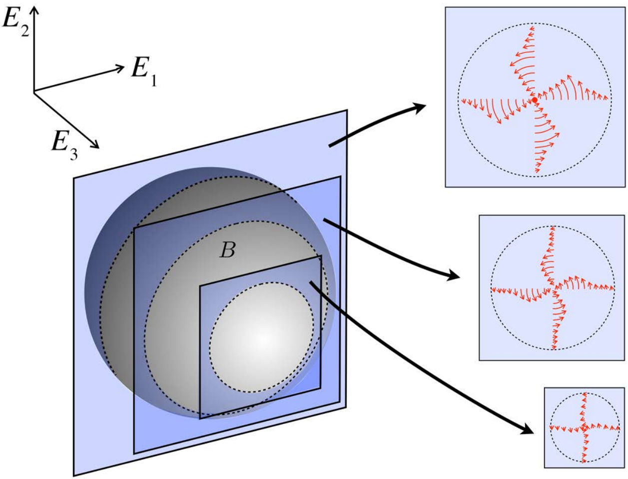
Remark

Origin of the method: Shub–Wilkinson, 2000.

Construction of the Baraviera–Bonatti perturbation

- Consider e.g. $d = 3$, $i = 1$.
- Take a small ball B centered at a non-periodic point.
- Perturb f inside B in a conservative way, approximately preserving and rotating the $E_1 \oplus E_2$ planes, obtaining some $\tilde{f}$.
  (See fig. next slide)
- Then one can show that the first two exponents "mix" a little (while the third almost doesn't move); in particular, $\lambda_1(\tilde{f}) < \lambda_1(f)$.

Rotating the $E_1 \oplus E_2$ planes

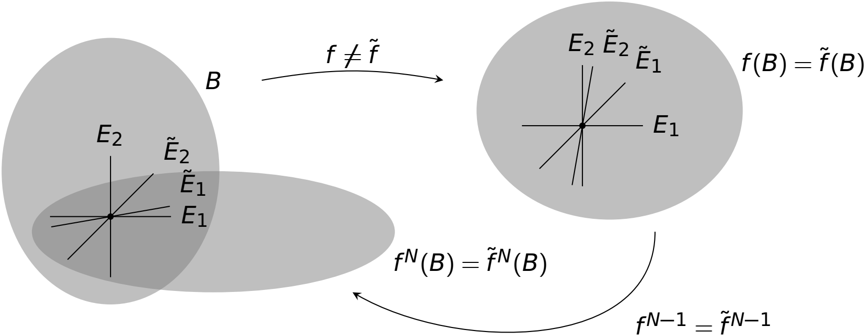


Rem: On each sphere concentric to ∂B the perturbation is a rotation.

Figure by Avila–Crovisier–Wilkinson

Why does λ_1 drop?

The new bundle $\tilde{E}_3$ is very close to the original E_3.
The other bundles move as follows:



So $N \gg 1 \Rightarrow \measuredangle(\tilde{E}_1, E_1) \ll 1$ on B.

Why does λ_1 drop? (continued)

To simplify notation, assume $T_xM = \mathbb{R}^d$, $E_i = \mathbb{R}e_i$.
Take nowhere-zero vector fields $v \equiv e_1$ and $\tilde{v}$ tangent to E_1 and $\tilde{E}_1$, respectively.
The seminorm $\|(a_1, \ldots, a_d)\|_1 := |a_1|$ is good enough to compute the first Lyapunov exponent:

$$\lambda_1(f) = \int_M \log \frac{\|Df(x)v(x)\|_1}{\|v(x)\|_1}\, dm(x)$$

$$\lambda_1(\tilde{f}) = \int_M \log \frac{\|D\tilde{f}(x)\tilde{v}(x)\|_1}{\|\tilde{v}(x)\|_1}\, dm(x)$$

The two integrands are everywhere equal, except on B.
On B we use that $\tilde{v} \simeq v$ to compare the integrals.
Jensen inequality $\Rightarrow \lambda_1(\tilde{f}) < \lambda_1(f)$.

Our proof

We rotate several $E_i \oplus E_{i+1}$ planes so to manipulate (i.e., “mix”) all the Lyapunov exponents simultaneously (careful Linear Algebra).

In order to maximize the effect of the Baraviera–Bonatti-like perturbations, it will be fundamental to use especially **adapted coordinates**.

A new adapted metric for dominated splitting

Given the simple dominated splitting $TM = E_1 \oplus \cdots \oplus E_d$ and a Riemannian norm $|||\cdot|||$, define *expansion functions* $\rho_1, \ldots, \rho_d \colon M \to \mathbb{R}$:

$$\rho_j(x) := \log \frac{|||Df(x)v|||}{|||v|||} \quad \text{(arbitrary nonzero } v \in E_j(x)\text{).}$$

Each ρ_j is continuous and its integral is $\lambda_j(f)$. We say that the Riemannian metric is *adapted* if:

$$\rho_1(x) > \rho_2(x) > \cdots > \rho_d(x) \quad \text{and} \quad E_i \perp E_j \ \forall i \neq j\,.$$

Proposition (Adapted metric with L^1 estimate)

Given $\varepsilon > 0$, we can choose an adpated metric such that $\int_M |\rho_i(x) - \lambda_i(f)| dm(x) < \varepsilon$ for every i.

Proof of existence of adapted metric with L^1-estimate

Since we are assuming simple dominated splitting, the situation becomes essentially one-dimensional.
The proof is a very simple and explicit averaging trick:

$$v \in E_j(x) \;\Rightarrow\; |||v||| := \prod_{n=0}^{N-1} \|Df^n(x)v\|^{1/N} \quad (N \gg 1)$$

Sketch of proof of the main theorem

We must be able to change (i.e., "mix") the Lyapunov spectrum $\vec{\lambda}(f)$ of f by some small but constant amount that depends **not on f but only on $\vec{\lambda}(f)$ itself.**

- We take a disjoint family of small "good" balls B_i (in the adapted coordinates) whose union has $N \gg 1$ disjoint iterates from itself (a tower).
- On each of these balls, we do Baraviera–Bonatti-like perturbations (rotating several planes).
- By Rokhlin Lemma, we can take $m\left(\bigsqcup B_i\right)$ approximately equal to $1/N$.

Sketch of proof of the main theorem (cont)

- Actually we will take height $\boxed{N \simeq C/\mathrm{GAP}}$, where $C \gg 1$ is fixed and $\mathrm{GAP} := \min_j \left[\lambda_j(f) - \lambda_{j+1}(f)\right]$. **Using the L^1 estimate for the adapted metrics**, we see that for most points, **time N is sufficient for cones to contract** and therefore for the Baraviera–Bonatti perturbation to have a controllable and significant effect on the Lyapunov exponents.
- More precisely, the effect on the Lyapunov exponents is approximately proportional to

$$m\left(\bigsqcup B_i\right) \sim \frac{1}{N} \sim O(\mathrm{GAP}).$$

- So we are able to change the Lyapunov spectrum by some small amount that depends **not on f but only on $\vec{\lambda}(f)$ itself.** Done!

The flexibility theorem on $\mathbb{T}^3$

In the situation of our flexibility theorem on $\mathbb{T}^3$, the starting Anosov diffeomophism is F_L. In particular, the invariant foliations are smooth. We can apply Baraviera–Bonatti preserving the (2-dim) center-unstable foliation (say) and therefore keeping $\boxed{\lambda_3(f_t) = \lambda_3(L)}$ along the deformation.

So we are able to realize spectra $\preccurlyeq \vec{\lambda}(L)$ (non-strict majorization).

(In large dimension it doesn't work so well. . .)

EXTENSIONS OF THE RESULTS?

Next results?

Our (upgraded Baraviera–Bonatti) method is very adaptable: being Anosov is not really important, but domination is.

Beyond PH/dominated systems, we should be able to allow domination to degenerate in a controlled way in a small "singular" set (like Katok'79, Dolgopyat–Pesin'02).

✎ So the general flexibility conjectures (arbitrary manifold) seem attackable, at least in some cases...

✎ Another direction: symplectic maps.