Exotic Lyapunov Spectra

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Cocycles

Our basic setting:

- X is a compact metric space;
- $f\colon X \to X$ is a continuous map (often a homeomorphism);
- $\mathrm{SL}^{\pm}(2,\mathbb{R})$ is the group of 2×2 real matrices with determinant ± 1;
- $A\colon X \to \mathrm{SL}^{\pm}(2,\mathbb{R})$ is a continuous map.

We call the pair (f, A) a *cocycle*.

We're interested in the matrix products:

$$A^{(n)}(x) := A(f^{n-1}x) \cdots A(fx)A(x)\,.$$

Uniform hyperbolicity

Let (f, A) be a cocycle (i.e. $f : X \to X$ and $A : X \to \mathrm{SL}^{\pm}(2, \mathbb{R})$ are continuous).

Definition

The cocycle (f, A) is *uniformly hyperbolic (UH)* if the norms of the products grow uniformly exponentially:

$$\exists \varepsilon > 0 \ \exists n_0 > 0 \ \forall x \in X \ \forall n \geq n_0 \ , \ \|A^{(n)}(x)\| > e^{\varepsilon n} \, .$$

Lyapunov exponents and spectrum

Let (f, A) be a cocycle. Let μ be an ergodic measure for f. By Oseledets' theorem, there exists $\lambda(A, \mu)$ such that for μ-a.e. x,

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)\| = \lambda(A, \mu)\,.$$

The *Lyapunov spectrum* of the cocycle (f, A) is the set

$$\Lambda(A) := \left\{\lambda(A, \mu)\,;\, \mu \text{ is ergodic}\right\}.$$

Uniform positivity of LE vs uniform hyperbolicity

If the cocycle (f, A) is UH, then the Lyapunov exponents of ergodic measures are away from 0:

$$\Lambda(A) \subset [\varepsilon, +\infty) \quad \text{for some } \varepsilon > 0.$$

The converse is false: take e.g. a NUH (nonuniformly hyperbolic) cocycle over an irrational rotation (Herman 1981).

Uniform positivity of LE vs uniform hyperbolicity (cont'd)

What if f has "lots" of invariant measures?

We say that a cocycle (f, A) is in the *Livsic setting* if:

- $f\colon X \to X$ is a hyperbolic homeomorphism (SFT or hyperbolic basic set);
- $A\colon X \to \mathrm{SL}^{\pm}(2, \mathbb{R})$ is a Hölder map.

Theorem (Velozo Ruiz 2020 (based on Cao–Luzzatto–Rios 2006))

There exists a cocycle in the Livsic setting such that the LE of all ergodic measures are uniformly away from zero, and yet the cocycle is not UH.

Idea: hyperbolicity degenerates along a single homoclinic orbit.

Sufficient conditions for UH

Theorem (Velozo Ruiz 2020)

If a cocycle (f, A) in the Livsic setting is fiber-bunched and the LE of all ergodic measures are uniformly away from zero, then it is UH.

Sufficient conditions for UH (cont'd)

Theorem (Guysinsky / DeWitt–Gogolev 2024)

If (f, A) is a cocycle in the Livsic setting and

$$\forall \text{ ergodic measure } \mu, \quad \lambda(A, \mu) = c\,,$$

where c is a positive constant, then the cocycle is UH. Actually, the hypothesis can be weakened to "narrow spectrum": all LE are "close" to c, where "closeness" depends on f and the Hölder exponent of A).

Example of point spectrum without UH

Our first result says that the Hölder hypothesis in the narrow spectrum theorem (Guysinsky / DeWitt–Gogolev) cannot be relaxed:

Theorem (Example 1)

There exists a hyperbolic homeomorphism $f : X \to X$ and a continuous map $A : X \to \mathrm{SL}^{\pm}(2, \mathbb{R})$ such that

$$\forall \text{ ergodic measure } \mu, \quad \lambda(A, \mu) = \text{some constant } c > 0\,,$$

but the cocycle (f, A) is not uniformly hyperbolic.

Can this be proved by tweaking Velozo Ruiz's example? I don't think so.

How does the LE depend on the measure?

Given a cocycle (f, A), the LE is a measurable function of the point which in general is **not** continuous.

The LE is **not** a continuous function of the measure either (unless the cocycle is UH...):

Example (a cocycle in the Livsic setting)

$$f = \text{ shift on } \{0,1\}^{\mathbb{Z}}$$

$$A = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} & \text{on the cylinder } [0] \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \text{on the cylinder } [1] \end{cases}$$

$\mu_n :=$ prob. measure supp'd on the periodic orbit $(0^n 1)^\infty$

$\mu :=$ prob. measure supp'd on the fixed pt. 0^∞

$$\Rightarrow \mu_n \xrightarrow{\text{weak } *} \nu \quad \text{but} \quad \underbrace{\lambda(A, \mu_n)}_{0} \not\to \underbrace{\lambda(A, \mu)}_{\log 2}$$

Approximation of LE using periodic orbits

Despite the previous difficulty, we have the following positive result:

Theorem (Kalinin 2011)

If (f, A) is a cocycle in the Livsic setting, and ν is any ergodic measure for f, then there exists a sequence of invariant measures μ_n supported on periodic orbits such that

$$\mu_n \xrightarrow{\text{weak } *} \nu \quad \text{and} \quad \lambda(A, \mu_n) \to \lambda(A, \mu)\,.$$

An example with no periodic approximation

Our second result says that the Hölder hypothesis in Kalinin's approximation theorem cannot be relaxed:

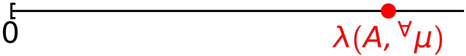
Theorem (Example 2)

Fix numbers $c > \varepsilon > 0$. There exists a hyperbolic homeomorphism $f\colon X \to X$ and a continuous map $A\colon X \to \mathrm{SL}^{\pm}(2,\mathbb{R})$ with the following properties:

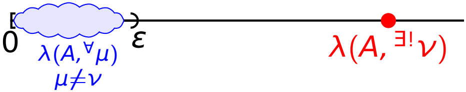
- *there exists an ergodic measure ν whose support is not a periodic orbit such that $\lambda(A,\nu) = c$;*
- *for all ergodic measures μ different from ν, we have $\lambda(A,\mu) \le \varepsilon$;*
- $\displaystyle\lim_{\substack{\mu\to\nu \\ \mu\neq\nu}} \lambda(A,\mu) = 0.$

Summary: the two examples

1. "Monochromatic" example:



2. "Bichromatic" example:



In each example:

- the base f is hyperbolic;
- the cocycle (f, A) is not UH;
- the cocycle map A is C^0 (and cannot be Hölder).

Strategy

The two constructions have some similarities:

- We start with a specific NUH cocycle over a strictly ergodic base dynamics.
- We embed this base dynamics in a hyperbolic dynamics f.
- We carefully extend the cocycle so that it has the desired properties.

Veech transformation

Theorem (Veech 1969)

There exists a compact metric space Y and a homeomorphism $f\colon Y \to Y$ such that:

- *f is strictly ergodic (i.e. minimal and uniquely ergodic);*
- *f^2 is minimal but **not** uniquely ergodic.*

f^2 admits exactly two ergodic measures ν_0, ν_1.

$$\operatorname{supp}(\nu_0) = \operatorname{supp}(\nu_1) = Y\,, \quad f_*(\nu_0) = \nu_1\,, \quad f_*(\nu_1) = \nu_0\,.$$

The unique f-invariant probability measure is:

$$\nu = \frac{\nu_0 + \nu_1}{2}\,,$$

Walters cocycle

$f\colon Y \to Y$ Veech map

$\nu =$ unique f-inv measure $= \frac{\nu_0+\nu_1}{2}$ where $f_*(\nu_i) = \nu_{1-i}$.

Choose a continuous function $\varphi\colon Y \to \mathbb{R}$ such that

$$\int \varphi \, d\nu_0 \neq \int \varphi \, d\nu_1 \, .$$

The *Walters cocycle* is (f, A) where

$$A(x) \coloneqq \begin{pmatrix} 0 & e^{\varphi(x)} \\ e^{-\varphi(x)} & 0 \end{pmatrix}$$

Note: $\det A \equiv -1$.

Proposition (Walters 1984)

The Walters cocycle has $\lambda(A, \nu) > 0$, but it is not UH.

Walters cocycle (cont'd)

Proof that Walters is NUH.

$$A(x) := \begin{pmatrix} 0 & e^{\varphi(x)} \\ e^{-\varphi(x)} & 0 \end{pmatrix} \quad \rightsquigarrow \quad A^{(2)}(x) = \begin{pmatrix} e^{\psi(x)} & 0 \\ 0 & e^{-\psi(x)} \end{pmatrix}$$

$$\psi := \varphi \circ f - \varphi$$

$$\int \psi \, d\nu_0 = \int \varphi \, d\nu_1 - \int \varphi \, d\nu_0 =: c \neq 0$$

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{j=0}^{n-1} \psi(f^{2j} x) \to \begin{cases} c & \text{for } \nu_0\text{-a.e. } x \\ -c & \text{for } \nu_1\text{-a.e. } x \end{cases}$$

$$\lambda(A, \nu) = |c| > 0$$

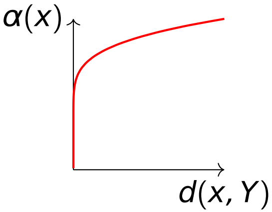
The Oseledets direction E^s is either $\mathbb{R}\binom{1}{0}$ or $\mathbb{R}\binom{0}{1}$, each case occurring on a dense set of ν-measure $\frac{1}{2}$. So the cocycle cannot be UH. $\square$

Construction of monochromatic cocycle

There is a Veech map that can be realized as a subshift $Y \subset \{1, 2, 3\}^{\mathbb{Z}}$.

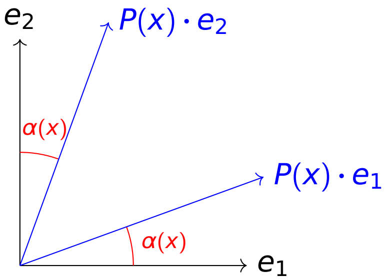
We extend the Walters cocycle to $X := \{1, 2, 3\}^{\mathbb{Z}}$, keeping the form $A(x) = \begin{pmatrix} 0 & e^{\varphi(x)} \\ e^{-\varphi(x)} & 0 \end{pmatrix}$.

We choose a continuous function $\alpha \colon X \to [0, \frac{1}{10}]$ such that $\alpha^{-1}(0) = Y$ but with a **very bad** modulus of continuity around Y:



Invariant cone

Take the following **positive** matrix $P(x)$ with determinant 1:



Define the (nonnegative) cocycle

$$B(x) := A(x)P(x)$$

Invariant field of directions

Lemma

There exists a continuous $u\colon X \setminus Y \to \mathbb{R}^2_+$ with $\|u(x)\| = 1$ such that

$$B(x)u(x) = \text{scalar} \cdot u(fx)\,.$$

Write the "scalar" as $e^{c+\rho(x)}$, where c is the LE of the Walters cocycle.

The function $\rho\colon X \setminus Y \to \mathbb{R}$ is continuous and bounded. If it were cohomologous to 0, the cocycle would be monochromatic, as desired.

Idea: Tweak the cocycle so that ρ becomes a coboundary.

Controlled times

Birkhoff sums:

$$S_n\rho(x) := \rho(x) + \rho(fx) + \cdots + \rho(f^{n-1}x)$$

Lemma (Technical core)

If the construction is done carefully, there exists a function $n\colon X \smallsetminus Y \to \mathbb{R}_+$ such that

- $n(x) \to \infty$ *"slowly" as* $x \to Y$.
- $\dfrac{S_{n(x)}\rho(x)}{n(x)} \to 0$ *as* $x \to Y$.

End of the construction

Lemma (Approximate cohomological equation)

For $x \in X \smallsetminus Y$,

$$\rho(x) = c + G(fx) - G(x) + R(x)$$

where G and R are continuous on $X \smallsetminus Y$ and

$$R(x) \to 0 \quad \text{as} \quad x \to Y \, .$$

Desired monochromatic cocycle:

$$\tilde{A}(x) := B(x)Q(x) \text{ where } \begin{cases} Q(x)u(x) = e^{-R(x)}u(x) \\ Q(x)u(x)^\perp = e^{R(x)}u(x)^\perp \end{cases}$$

for $x \in X \smallsetminus Y$, and $\tilde{A} =$ the Walters cocycle on Y. ∎

The second theorem (again)

Theorem (Bichromatic example)

Fix numbers $c > \varepsilon > 0$. There exists a hyperbolic homeomorphism $f\colon X \to X$ and a continuous map $A\colon X \to \mathrm{SL}^{\pm}(2,\mathbb{R})$ with the following properties:

- *there exists an ergodic measure ν whose support is not a periodic orbit such that $\lambda(A,\nu) = c$;*
- *for all ergodic measures μ different from ν, we have $\lambda(A,\mu) \le \varepsilon$;*
- $\displaystyle\lim_{\substack{\mu\to\nu\\ \mu\neq\nu}} \lambda(A,\mu) = 0.$

Fibonacci word

Define a sequence of words F_n on letters 0, 1 by the rules:

$$F_0 = 0, \quad F_1 = 01, \quad F_{n+2} = F_{n+1}F_n$$

So:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 01 \\ F_2 &= 010 \\ F_3 &= 01001 \quad \text{etc.} \end{aligned}$$

Let F_∞ be the unique infinite word that has each F_n as a prefix.

Fibonacci in the circle

The letters of F_∞ are the binary digits of a number:

$$\xi := 0.01001010\ldots_{\text{base two}} = 0.290196557\ldots_{\text{base ten}}$$

Let f be the doubling map on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

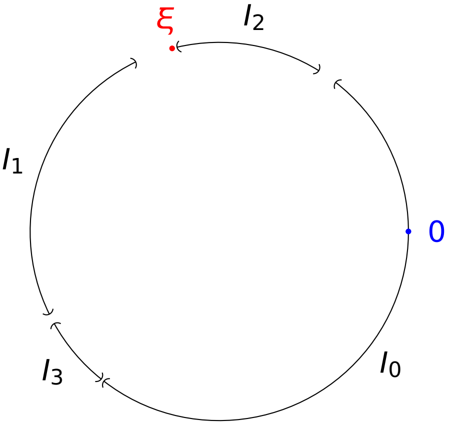
Let K be the closure of the f-orbit of ξ. This is a (zero-dimensional) Cantor set that contains ξ.

The map $f|_K$ is a homeomorphism, and it is semiconjugate to the irrational rotation by $\alpha := \frac{3-\sqrt{5}}{2}$.

Gap structure

The gaps of the Cantor set K can be enumerated as intervals $I_0, I_1, I_2, \ldots$ of respective lengths $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$$f(I_{n+1}) = I_n, \quad f(I_0) = \mathbb{T} \setminus \{\xi\}$$



An initial cocycle

Choose a NUH cocycle (R_α, B) over the (irrational) rotation by $\alpha = (3 - \sqrt{5})/2$.

Using the semiconjugacy $h\colon K \to \mathbb{T}$, this cocycle can be realized as $(f|_K, A)$, for some $A\colon K \to \mathrm{SL}(2, \mathbb{R})$:

$$A(x) = B(h(x)), \quad x \in K$$

Remark: Note that if p, q are the extremes of a same interval I_n, then $A(p) = A(q)$.

Set that aside while we unpack the next ingredient...

Accessibility by bounded cocycles

Theorem (Avila–B.–Damanik 2012)

Let g be an irrational rotation (or essentially any non-periodic stricly ergodic homeo) and let $B\colon \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$ be homotopic to constant. If the cocycle (g, B) is not UH, then there exists an homotopy $(B_t)_{t\in[0,1]}$ from $B_0 = B$ to $B_1 = \text{const}\ \mathrm{Id}$ such that for each $t \in (0,1]$, the cocycle (g, B_t) has zero LE, and actually bounded products:

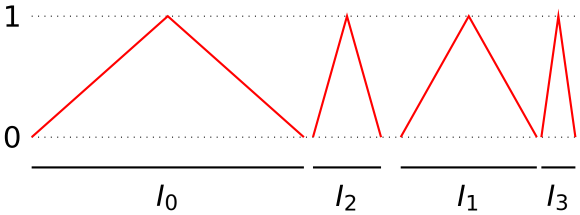
$$M(t) := \sup_n \sup_x \|B_t^{(n)}(x)\| < \infty\,.$$

Rem: Furthermore, $M(\cdot)$ is continuous on $(0,1]$. By reparametrization, we can assume that $M(t) \to 0$ **slowly** as $t \to 0$.

Warm-up: a discontinuous example

We already defined a NUH cocycle on the Fibonacci Cantor set $K \subset \mathbb{T}$ as $A(x) := B(h(x))$; we want to extend it to the circle $\mathbb{T}$.

Define a (discontinuous) function $\rho \colon \mathbb{T} \to [0, 1]$:



Use ρ to "insert" the ABD homotopy in each gap:

$$x \in I_k \quad \Rightarrow \quad A(x) := B_t(h(p)) \text{ with } t = \rho(x)\,, p \in \partial I_k\,.$$

Warm-up: a discontinuous example (cont'd)

Claim: $\lim_{\substack{\mu\to\nu \\ \mu\neq\nu}} \lambda(A,\mu) = 0.$ $\quad$ ($\nu :=$ unique f-inv. meas. supp. K).

Sketch of proof:

- If μ is weak-*-close to ν, then $\mu(I_0)$ is close to 0.
- By the Kac formula, for μ-typical points x, the return times n_i to I_0 are large (in average).
- So the orbit of x traverses long towers $I_{n_i} \to I_{n_i-1} \to \cdots \to I_0$.
- **The tent geometry is preserved under the doubling map**, so the corresponding cocycle product involves only B_t's with the same $t = t_i$, and therefore is uniformly bounded by some $M(t)$.
- Can we bound $\frac{\log M(t_i)}{n_i}$? Not directly, but...

Warm-up: a discontinuous example (cont'dd)

- If t is very small (which is bad), then the (return) point $y \in I_0$ is very close to ∂I_0, so $f(y)$ is very close to ξ.
- So the *next* return time n_{i+1} is big (which is good), so ultimately $\log M(t_i) \le F(n_{i+1})$ for some function F.
- Reparametrizing the homotopy we can assume F is sublinear (e.g. $F(n) = \sqrt{n}$).
- $\lambda(A, x) \le \limsup_{i \to \infty} \dfrac{F(n_2) + F(n_3) + \cdots + F(n_{i+1})}{n_1 + n_2 + \cdots + n_i}$ is small.

Construction of the bichromatic example (finale)

Choose a sequence of times scales

$$0 = \ell_0 \ll \ell_1 \ll \ell_2 \ll \cdots .$$

For each n in the interval $[\ell_k, \ell_{k+1})$, squish the tent on I_n by factor $\frac{1}{k}$.

With this modification, the cocycle is now continous.

The "tent geometry" is still preserved on long pieces of orbits.

With more work we can bound the Lyapunov exponent.

■

Improving the second theorem?

Question

Does there exist a truly “bichromatic” example? (Lyapunov exponent **exactly** zero for all ergodic measures except one).

Question

Can we do even better and make the “periodic data” trivial?

$$f^n(p) = p \quad \Rightarrow \quad A^{(n)}(p) = \mathrm{Id}$$