ON THE JOINT MAXIMAL DRIFT OF EUCLIDEAN ISOMETRIES

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Abstract. Given a finite family of isometries (or, more generally, semicontracions) of a noncompact metric space, its *joint maximal drift* is the maximal escape speed of compositions of their elements in arbitrary order. The shift-invariant measures that attain this maximal speed are called *drift-maximizing measures*. In the case of isometries of the hyperbolic plane, the joint maximal drift and the drift-maximizing measures are relatively understood, because there are direct connections with the theory of the joint spectral radius. This paper deals with isometries of Euclidean space. In this setting, drift-maximizing measures are typically *not* supported on periodic orbits and the joint maximal drift is *not* continuous, in strong contrast with the hyperbolic panorama. Among various results, we show that on a subset of parameter space with positive Lebesgue measure (related to specific Diophantine properties), the drift is maximized along appropriate quasiperiodic compositions of the isometries: the corresponding drift-maximizing measures are Sturmian. Our study extends the theory of ergodic optimization into new territory. An important step in our reasoning is a reduction to optimization of ergodic averages with respect to a specific skew-product dynamical system. However, this system is only partially hyperbolic, and thus outside the scope of the previously available theory.

Contents

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1. Introduction

1.1. **Joint maximal drift.** Karlsson [Kar22] has put forward a program to extend concepts and results from linear functional analysis by "forgetting the linearity of the space and instead focusing on merely the metric structure". This approach often leads to surprisingly powerful and elegant results, for instance the Gouëzel–Karlsson multiplicative ergodic theorem [GK20]. In Karlsson's program, linear operators correspond to semicontractions and the metric analog of the spectral radius of a linear operator is the drift of a semicontraction. We proceed with the corresponding definitions.

Let (M, d) be a metric space, which for all purposes can be assumed to be unbounded. A map $f \colon M \to M$ is called a *semicontraction* if it is 1-Lipschitz, i.e.,

$$\mathrm{d}(f(x), f(y)) \leqslant \mathrm{d}(x, y) \quad \text{for all } x, y \in X\,. \tag{1.1}$$

In many situations of interest, f is actually an isometry. If f is a semicontraction, then for any *basepoint* $x_0 \in M$, the sequence $(\mathrm{d}(f^n(x_0), x_0))_{n \geqslant 0}$ is subadditive:

$$\begin{aligned}\mathrm{d}(f^{n+m}(x_0), x_0) &\leqslant \mathrm{d}(f^{n+m}(x_0), f^m(x_0)) + \mathrm{d}(f^m(x_0), x_0) & (1.2)\\ &\leqslant \mathrm{d}(f^n(x_0), x_0) + \mathrm{d}(f^m(x_0), x_0)\,. & (1.3)\end{aligned}$$

Therefore, by Fekete's lemma, the following limit exists:

$$\mathrm{Dr}(f) \coloneqq \lim_{n \to \infty} \frac{\mathrm{d}(f^n(x_0), x_0)}{n}\,. \tag{1.4}$$

If $\tilde{x}_0 \in M$ is another basepoint, then another application of the semicontraction property shows that, for any $n \geqslant 0$,

$$|\mathrm{d}(f^n(\tilde{x}_0), \tilde{x}_0) - \mathrm{d}(f^n(x_0), x_0)| \leqslant 2\mathrm{d}(\tilde{x}_0, x_0)\,. \tag{1.5}$$

It particular, the limit $\mathrm{Dr}(f)$ is independent of the choice of the basepoint. We call this quantity the *drift* of the semicontaction f. (Other terminology include *escape rate*, *translation number*, or *asymptotic displacement.*) The similarity between the definition (1.4) and Beurling–Gelfand formula $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ for the spectral radius of a linear operator is not a trifling observation: it actually has profound repercussions, as discussed in [Kar22, § 5].

In their (belatedly) celebrated 1960 paper [RS60], Rota and Strang defined the *joint spectral radius* of a bounded family $\mathcal{A}$ of linear operators as

$$\mathrm{JSR}(\mathcal{A}) \coloneqq \lim_{n \to \infty} \max \left\{ \|A_{i_1} \dots A_{i_n}\|^{\frac{1}{n}} : A_{i_1}, \dots, A_{i_n} \in \mathcal{A} \right\}. \tag{1.6}$$

We propose adding an entry to Karlsson's dictionary by defining the appropriate metric analog of the joint spectral radius, which will be the joint maximal drift of a family of semicontractions. Before giving the definition, we make a couple of remarks. To reduce technicalities, we will consider only finite families, though much of what we will say can be adapted to infinite families under appropriate assumptions. Furthermore, it is convenient to formulate our concepts in terms of representations.[1]

In all of this paper, let $\mathcal{S}$ be a semigroup freely generated by a finite set $\mathcal{A}$. The elements of $\mathcal{A}$ and $\mathcal{S}$ are called *letters* and *words*, respectively. The word-length of an element $w \in \mathcal{S}$ with respect to the set of generators is denoted $|w|$.

[1] Other approaches are iterated function systems and switched dynamical systems.

Denote by $\mathrm{Semi}(M)$ the semigroup of semicontractions of the metric space (M, d). Let $\rho\colon \mathcal{S} \to \mathrm{Semi}(M)$ be a representation, that is, a semigroup morphism. The *joint maximal drift* of the representation ρ is defined as:

$$\mathrm{JMD}(\rho) \coloneqq \lim_{n\to\infty} \frac{1}{n} \max\left\{ \mathrm{d}(\rho(w)(x_0), x_0) : w \in \mathcal{S},\ |w| = n \right\}; \tag{1.7}$$

similarly to (1.4), the limit exists and is independent of the choice of the basepoint $x_0 \in M$. This quantity was introduced under the name *joint stable length* by Reyes [OR18], in the setting of isometries of Gromov hyperbolic spaces. Together with other related quantities and in more general settings, it was further studied by Breuillard and Fujiwara [BF21]; their terminology is *asymptotic joint displacement.*

As an example, suppose M is the hyperbolic plane $\mathbb{H}^2$, represented as the upper half plane. Then $\mathrm{Isom}(\mathbb{H}^2)$ is isomorphic to the group of 2×2 matrices of determinant ± 1 and $\mathrm{JMD}(\rho)$ equals twice the log of the joint spectral radius of the set of matrices associated to the isometries $\rho(a)$, $a \in \mathcal{A}$; see [OR18, Proposition 2.1.(iii)].

1.2. **Euclidean isometries.**

Though the defintion of joint maximal drift applies to representations by semicontractions of any metric space, in this paper we actually work with representations by isometries of Euclidean space. Let $\|x\|$ denote the Euclidean norm of a vector $x \in \mathbb{R}^d$. The joint maximal drift of a representation $\rho\colon \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$ can be expressed as follows, choosing the origin as a basepoint in (1.7):

$$\mathrm{JMD}(\rho) \coloneqq \lim_{n\to\infty} \frac{1}{n} \max\left\{ \|\rho(w)(0)\| : w \in \mathcal{S},\ |w| = n \right\}. \tag{1.8}$$

The case of $\mathrm{JMD}(\rho) = 0$ admits the following simple description:

Proposition 1.1 (Breuillard and Fujiwara [BF21, Proposition 9.3])**.** Let $\rho\colon \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$ be a representation by Euclidean isometries. Then $\mathrm{JMD}(\rho) = 0$ if and only if the isometries $\rho(w)$, $w \in \mathcal{S}$ admit a common fixed point.

Note that the "if" part of the proposition is trivial and holds for representations $\rho\colon \mathcal{S} \to \mathrm{Semi}(M)$ where M is arbitrary metric space. On the other hand, the "only if" part fails for representations by isometries of hyperbolic space, since parabolic isometries have zero drift but no fixed points. Some infinite-dimensional Banach spaces also admit isometries with zero drift and no fixed points [Nav25].

1.3. **Ergodic viewpoint.**

We provide an interpretation of the joint maximal drift in terms of ergodic theory.

Suppose X is a compact metric space and $T\colon X \to X$ is a continuous map. Let $\mathcal{M}_T$ denote the set of all T-invariant probability measures, endowed with the weak topology. Then $\mathcal{M}_T$ is a nonempty compact convex set. Now let $\Phi = (\varphi_n)$ be a sequence of upper semicontinuous functions $\varphi_n\colon X \to [-\infty, \infty)$. Assume that the sequence is *subadditive*, that is,

$$\varphi_{m+n} \leqslant \varphi_m \circ T^n + \varphi_m \quad \text{for all } m, n > 0\,. \tag{1.9}$$

By Kingman's subadditive ergodic theorem, there exists a Borel set $B \subseteq X$ such that $\mu(B) = 1$ for every $\mu \in \mathcal{M}_T$ and there exists a Borel function $\bar{\varphi}\colon X \to [-\infty, \infty)$ such that

$$\lim_{n\to\infty} \frac{\varphi_n(x)}{n} = \bar{\varphi}(x) \quad \text{for every } x \in B\,. \tag{1.10}$$

We define the *ergodic average* of the sequence Φ with respect to a measure $\mu \in \mathcal{M}_T$ as

$$\mathrm{A}(\Phi, \mu) := \int \bar{\varphi}\, d\mu\,. \tag{1.11}$$

Thus, if μ is ergodic, then $\frac{\varphi_n(x)}{n} \to \mathrm{A}(\Phi, \mu)$ for μ-almost every x.

The *maximal ergodic average* of the sequence $\Phi = (\varphi_n)$ is defined as

$$\mathrm{MA}(\Phi) := \sup_{\mu \in \mathcal{M}_T} \int \bar{\varphi}\, d\mu\,. \tag{1.12}$$

It admits the following alternative descriptions [Mor13, Theorem A.3]:

$$\mathrm{MA}(\Phi) = \lim_{n\to\infty} \sup_{x \in X} \frac{\varphi_n(x)}{n} \tag{1.13}$$

$$= \sup_{x\in X} \inf_n \frac{\varphi_n(x)}{n}\,. \tag{1.14}$$

where the limit in (1.13) is also an inf (by Fekete's lemma).

Any $\mu \in \mathcal{M}_T$ that attains the supremum in (1.12) is called a *maximizing measure*. The set of maximizing measures is a nonempty compact convex subset of $\mathcal{M}_T$ and, in particular, it contains ergodic elements.

Let $\Sigma := \mathcal{A}^{\mathbb{N}}$ be space of sequences of letters (where $\mathbb{N}$ is the set of positive integers, endowed with the product topology, and let $\sigma \colon \Sigma \to \Sigma$ be the shift map. We denote the i-th letter of $\omega \in \Sigma$ by $\omega[i]$, while if $1 \leqslant i \leqslant j$, then $\omega[i \colon j]$ denotes the word $\omega[i] \cdots \omega[j]$, an element of the semigroup $\mathcal{S}$.

If $\rho \colon \mathcal{S} \to \mathrm{Semi}(M)$ is a representation by semicontractions of a metric space $M = (M, d)$ and $x_0 \in M$ is a basepoint, then we define a sequence of continuous functions $\varphi_n \colon \Sigma \to [0, \infty)$ by

$$\varphi_n(\omega) := d\big(\rho(\omega[1 \colon n])x_0, x_0\big)\,. \tag{1.15}$$

Then the sequence $\Phi = (\varphi_n)$ is subadditive. In particular, for any $\mu \in \mathcal{M}_\sigma$, the ergodic average $\mathrm{A}(\Phi, \mu)$ is well-defined. This quantity is called the *drift* of the representation ρ with respect to the measure μ; explicitly,

$$\mathrm{Dr}(\rho, \mu) := \lim_{n\to\infty} \frac{1}{n} \int d\big(\rho(\omega[1 \colon n])x_0, x_0\big)\, d\mu(\omega)\,. \tag{1.16}$$

It was shown by Karlsson and Margulis [KM99] that if μ is ergodic and M has non-positive curvature in the sense of Busemann, then the sequence of points $\rho(\omega[1 \colon n])x_0$ is sublinearly "tracked" by a geodesic ray with speed $\mathrm{Dr}(\rho, \mu)$. This statement essentially contains the Multiplicative Ergodic Theorem of Oseledets. An even more powerful result was obtained later by Gouëzel and Karlsson [GK20], who showed that the escaping of sequence of points is "directed" by a metric functional.

In this setting, the maximal ergodic average coincides with the joint maximal drift, that is,

$$\mathrm{MA}(\Phi) = \mathrm{JMD}(\rho)\,, \tag{1.17}$$

as a direct consequence of (1.13). Using the equivalent characterization (1.12), we obtain

$$\mathrm{JMD}(\rho) = \sup_{\mu \in \mathcal{M}_\sigma} \mathrm{Dr}(\rho, \mu)\,. \tag{1.18}$$

Any measure μ attaining the supremum is called a *drift-maximizing measure*.

Back to the context of Euclidean isometries, we observe that their random (i.i.d.) composition was studied by several authors. Under mild conditions on $\rho\colon \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$, the drift of any Bernoulli measure is zero, and the distribution of the points $\rho(\omega[1\colon n])x_0$ satisfies a central limit theorem [Gor73, ÅMN03], local limit theorems [Var15, DH26], and a large deviation principle [Cor26]. In particular, drift-maximizing measures are not Bernoulli measures, except in very particular circumstances.

1.4. **Breakdown of periodic maximization.** Given a topological dynamical system (X, T) and a subadditive sequence $\Phi = (\varphi_n)$ of upper semicontinuous functions, we defined a maximizing measure as one that attains the supremum in (1.12). The study of maximizing measures is called *ergodic optimization*. Though this terminology is sometimes restricted to the case of *additive* sequences (those for which inequality (1.9) becomes an equality), it is clearly beneficial to allow subadditive sequences as well.

Ergodic optimization originated with the works [CG93, HO96, Jen96, YH99, Bou00, CLT01, BM02] and took off from there. General expositions on the subject can be found in [Jen06, BLL13, Gar17, Boc18, Jen19]. Recent important advances include [Mor21, HLM+25, GS24, GSZ25, HJXZ26]. The field is naturally connected to Lagrangian dynamics (action minimizing measures) and thermodynamical formalism (zero temperature limits).

The *typical periodic optimization* phenomenon is significant in ergodic optimization. It roughly means that, for sufficiently "chaotic" dynamics, maximizing measures of "typical" sufficiently regular functions should be supported on periodic orbits. More information can be found in several of the aforementioned references. An important theorem in this direction was proved by Contreras [Con16]. The strongest result obtained so far is due to Gao, Shen, and Zhang [GSZ25], who showed that if the base dynamics is an analytic expanding map of the circle, typical periodic optimization holds in any function space C^r, $1 \leqslant r \leqslant \omega$, both in the topological and the probabilistic senses of typicality. Beyond the additive setting, fewer results about typical periodic optimization are known. For the joint spectral radius, it is conjectured that maximizing measures are typically supported on periodic orbits [Mae08, Conjecture 8]. This conjecture was verified only in some particular configurations [MS13, Las25, Gao25].

Back to the context of representations by Euclidean isometries and maximization of the drift, let us see that periodic optimization is *not* typical in general. To begin, we observe the following fact:

Proposition 1.2. Suppose the representation $\rho\colon \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$ admits no common fixed point nor a word $w \in \mathcal{S}$ such that $\rho'(w)$ has an eigenvalue 1. Then no drift-maximizing measure can be supported on a periodic orbit.

Proof. By Proposition 1.1, $\mathrm{JMD}(\rho) > 0$. Let μ be supported on the orbit of a periodic point $\omega \in \Sigma$ of period n. The isometry $\rho(\omega[1\colon n])$ has a fixed point, because 1 is not an eigenvalue of its linear part. So μ has zero drift and in particular is not drift-maximizing. □

If d is even and $\#\mathcal{A} \geqslant 2$, the hypotheses of the proposition are satisfied on a full measure G_δ subset of the space of orientation-preserving representations.

Therefore, typical periodic optimization fails for orientation-preserving isometries of even-dimensional Euclidean space.[2]

A related phenomenon, the *failure of the Berger-Wang property*, was observed by Breuillard and Fujiwara [BF21, Proposition 9.4]. Wer also note that the *locking property*, which is ubiquitous in additive ergodic optimization, is not necessarily so in the context of drift-maximizing measures for Euclidean isometries: we discuss this issue in Subsection 3.4.

To the best of our knowledge, the only other known instance where typical periodic optimization fails despite an abundance of periodic orbits is the recent [HJXZ26, Theorem 1.2].

1.5. **Determining drift-maximizing measures.** Ideally, we would like to determine all drift-maximizing measures of a given representation by Euclidean isometries. In the case where the linear parts of the representation are contained in a finite subgroup of the orthogonal group $\mathrm{O}(d)$, this problem can be completely solved: unless there is a common fixed point, there are at most finitely many ergodic drift-maximizing measures, and all of them are supported on periodic orbits. Furthermore, the list of "candidates" is finite, and thus the joint maximal drift can be computed in finite (but not necessarily short) time. These are essentially the contents of our Theorem 3.1 below.

If the linear parts of the representation are not contained in a finite group, identifying the drift-maximizing measures is a much more delicate problem. Nevertheless, we manage to solve this problem in some cases. We consider the simplest possible non-trivial setting, namely, of a representation generated by only two orientation-preserving isometries of $\mathbb{R}^2$. Such a representation can be specified by six real parameters, but after discarding trivial cases, we see that the pair (α_1, α_2) of rotation angles is sufficient to determine the maximizing measures, and the relevant parameter space becomes the square $[-\pi, \pi]^2$. We explicitly describe a subset $\mathfrak{S}_0$ of parameter space where the unique drift-maximizing measure is a Sturmian measure with slope $|\alpha_1|/(|\alpha_1| + |\alpha_2|)$. In particular, the joint maximal drift can be computed exactly for all parameters in the set $\mathfrak{S}_0$. The definition of $\mathfrak{S}_0$ requires a Diophantine-like condition on the slope. In particular, the set $\mathfrak{S}_0$ is not open. This is not a shortcoming of our proof, as we prove that the set of parameters for which the Sturmian measure is maximizing is in fact nowhere dense in the two-torus. These statements are contained in Theorem 4.1 below. An approximate picture of the set $\mathfrak{S}_0$ can be found in Figure 9.

1.6. **Continuity and discontinuity of the joint maximal drift.** If the semigroup $\mathcal{S}$ and the dimension d are fixed, the space of all representations $\rho\colon \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$ can be identified with the product of finitely many copies of the isometry group. The joint maximal drift $\mathrm{JMD}(\rho)$ then becomes a function on this space. Exact computation of the values of function can be a non-trivial task, as the results mentioned above indicate. Therefore, we take a step back and investigate general properties of the JMD function.

The JMD is not a continuous function. However, it is upper semicontinuous (Proposition 5.1 below) and, as such, its set of points of continuity is a residual (dense G_δ set). We go beyond this observation and prove that if the linear parts of

[2]The picture changes if $d = 2$ but orientation-reversing isometries are allowed, as shown by Example 3.8. What happens in higher dimensions is unclear.

the representation ρ generate a semigroup which is dense in either $\mathrm{O}(d)$ or $\mathrm{SO}(d)$, then ρ is a continuity point of the joint maximal drift (Theorem 5.4). Furthermore, we prove that this criterion for continuity is sharp if $d = 2$ and all isometries preserve orientation (Theorem 6.1).

1.7. **Discussion of the methods.** Given a representation $\rho\colon \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$, the set of all of its "trajectories" has a well-defined asymptotic shape. More precisely, given a basepoint $x_0 \in \mathbb{R}^d$, for each $n > 0$ we consider the set of the points $\rho(w)(x_0)$, where w runs on all words of length n (with respect to the generating set $\mathcal{A}$). We prove that this sequence of sets, if scaled by a factor $1/n$, converges with respect to the Hausdorff metric to a compact convex set $J(\rho)$, which is independent of the choice of the basepoint (Theorem 2.3), and which we call *joint displacement set*. This set is just a ball of radius $\mathrm{JMD}(\rho)$ centered at the origin for typical representations ρ. On the other hand, for representations whose linear parts are contained in a finite group, the set $J(\rho)$ is a polytope. The joint displacement set is also an interesting object by itself, but our main reason for defining it is that it facilitates the study of the joint maximal drift.

While the joint maximal drift naturally pertains to *subadditive* ergodic optimization, nearly all of our proofs rely on the reduction to an *additive* setting.

We have already introduced the shift map σ on the space $\Sigma \coloneqq \mathcal{A}^{\mathbb{N}}$. Given a representation $\rho\colon \mathcal{S} \to \mathrm{Isom}(\mathbb{R}^d)$, we define a transformation T_ρ of the product space $\Sigma \times \mathrm{O}(d)$ which is a skew-product over the shift and acts on fibers by right-multiplication by the linear part of the corresponding isometry $\rho(\omega[1])$: see definition (2.13). (This map has been considered before [NMA01, ÅMN03].) It turns out that the objects we are interested in admit reformulations involving the skew-product T_ρ.[3] In fact, the joint displacement set becomes a set of integrals with respect to T_ρ-invariant measures (Theorem 2.6) and the joint maximal drift of ρ can be expressed as the maximal ergodic average of an additive sequence with respect to T_ρ (formula (2.45)).

The skew-product T_ρ can be considered as special type of partially hyperbolic dynamics, with isometric fibers, and thus part of a family that includes the classical frame flow [BP74]. While some authors have studied ergodic optimization beyond the uniformly hyperbolic setting, systems such as T_ρ have not been considered so far. In the absence of a general theory, our identification of maximizing measures (such as the Sturmian measures mentioned above) is somewhat ad hoc.

1.8. **Dependence between sections.** Section 2 contains basic results that are used throughout the paper. Sections 3, 4, and 5 are essentially independent, except that we appeal to Theorem 5.4 from Section 5 at the end of Section 4 and in the beginning of Section 6. The final Section 7 discusses directions for future investigation.

2. Joint displacement set and reduction to additive ergodic optimization

2.1. **Linear part and translation part.** It is a well-known elementary fact that every isometry of the Euclidean space $\mathbb{R}^d$ is an affine map, actually the composite of an ortogonal transformation and a translation [Ree83, p. 9]. Therefore, the isometry group $\mathrm{Isom}(\mathbb{R}^d)$ is a semidirect product $\mathrm{O}(d) \ltimes \mathbb{R}^d$, where $\mathrm{O}(d)$ is the orthogonal

[3] There is a technical caveat: it may be necessary to replace the fiber $\mathrm{O}(d)$ by a subgroup G_ρ.

The complete version of the file is available under request.

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