Emergences in Ergodic Theory

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joint work with Pierre Berger (CNRS)

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arXiv.org > math > **arXiv:1901.03300**

Mathematics > Dynamical Systems

On Emergence and Complexity of Ergodic Decompositions

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Introduction

Quantifying the complexity of a dynamical system:

- Entropy (either topological or metric) measures how many orbits are there (up to a given small resolution, up to a given long time frame). It may be that all (or most) of these orbits obey the same statistics.
- Emergence measures how many statistics are there...

"One of the most seductive buzzwords of complexity science is 'emergence'. Arguments rage about what it means (…) To me, emergence means non-unique statistical behaviour."

– Robert S. MacKay, Nonlinearity in complexity science, 2008.

“Note that emergence is very different from chaos, in which sensitive dependence produces highly non-unique trajectories according to their initial conditions. Indeed, the nicest forms of chaos produce unique statistical behaviors in the basin of the attractor. The distinction is like that between the weather and the climate. For weather we care about individual realizations; for climate we care about statistical averages.”

– Robert S. MacKay, Nonlinearity in complexity science, 2008.

Part 1:
Discretization of spaces of measures

HOW TO DISCRETIZE A METRIC SPACE

(Well-known) def. A metric space (X, d) is totally bounded if $\forall \varepsilon > 0$, X can be covered by finitely many (say, closed) balls of radius ε.

Equivalenty: $\forall \varepsilon > 0$ $\exists$ a finite ε-dense subset.

Def. The covering number $\boxed{D(\varepsilon) = D_{X,d}(\varepsilon)}$ is the minimum cardinality of such a cover/subset.

"D" stands for dense.

Def. The packing number $\boxed{S(\varepsilon) = S_{X,d}(\varepsilon)}$ is the maximum cardinality of a ε-separated subset.

"S" stands for separated

$F \subseteq X$ s.t. $d(x,y) > \varepsilon \ \forall x \neq y \in F$

(Well-known and easy) fact: $\boxed{S(2\varepsilon) \underset{(1)}{\leq} D(\varepsilon) \underset{(2)}{\leq} S(\varepsilon).}$

(1) A 2ε-separated set of cardinality N cannot be covered by less than N closed balls of radius ε.

(2) Every maximal ε-separated set is ε-dense.

Def. Box-counting dimension of (X,d)

$$\boxed{\dim(X) := \lim_{\varepsilon \to 0} \frac{\log D(\varepsilon)}{-\log(\varepsilon)}} \in [0,\infty]$$

I.e. $D(\varepsilon)$ is roughly $(1/\varepsilon)^{\dim(X)}$ ("polynomial")

- Lower / Upper box dimensions $\underline{\dim}(X) \leq \overline{\dim}(X)$: use liminf / limsup instead of lim.
- It makes no difference if we use the packing number $S(\varepsilon)$ instead of $D(\varepsilon)$.

What about if (X,d) is infinite dimensional?

It turns out that there are many interesting situations where $D(\varepsilon)$ (or $S(\varepsilon)$) is roughly a stretched exponential $\boxed{e^{(1/\varepsilon)^{\theta}}}$.

The corresponding exponent θ is called the metric order of (X,d). Formally:

$$\boxed{\mathrm{mo}(X) := \lim_{\varepsilon \to 0} \frac{\log \log D(\varepsilon)}{-\log(\varepsilon)}}.$$

- Lower / Upper metric orders $\underline{\mathrm{mo}}(X) \leq \overline{\mathrm{mo}}(X)$: use liminf / limsup.

The terminology "metric order" comes from Kolmogorov, Tihomirov, 1961

ε-ENTROPY AND ε-CAPACITY OF SETS IN FUNCTIONAL SPACES

A. N. KOLMOGOROV AND V. M. TIHOMIROV

CONTENTS

They give precise estimates for covering and packing numbers for several (infinite-dimensional) spaces of functions (Lipschitz, C^r, analytic) obeying given bounds...

A nice example of an ∞-dimensional compact space:

$$\boxed{M(X) := \{\text{probability Borel measures on } X\}},$$

endowed with the weak-$*$ topology.

Here X is a compact metric space.

(or, more generally, a totally bounded Borel space)

"Natural" ways of metrizing $(\mathcal{M}(X), \text{weak-}*)$

$\begin{cases} \bullet \text{ Lévy - Prokhorov metric } \quad LP \\ \bullet \text{ Wasserstein metrics } \quad W_p, \quad p \in [1, \infty) \end{cases}$

→ Both metrics are induced by the metric d on X.

→ They are such that X embeds isometrically on $\mathcal{M}(X)$ as follows: $x \in X \longmapsto \delta_x \in \mathcal{M}(X)$.

Levy – Prokhorov metric LP

$LP(\mu_1, \mu_2) :=$ least $\varepsilon \geq 0$ such that

$\forall$ closed set $C \subseteq X$, $\forall (i,j) = (1,2), (2,1)$

$\mu_i(\varepsilon\text{-neighborhood of } C) \geq \mu_j(C) - \varepsilon.$

Wasserstein metrics W_p, $1 \le p < \infty$.

$W_p(\mu_1, \mu_2) :=$ inf p-cost of a "transport plan" from μ_1 to μ_2.

Example: If $\exists$ measurable map $T : X \to X$ such that $T_* \mu_1 = \mu_2$ then T gives a transport plan whose p-cost is:

$$\left(\int_X d(x, T(x))^p \, d\mu(x) \right)^{1/p}.$$

Theorem*. Let (X,d) be compact (or totally bdd., Baire)

Metrize $\mathcal{M}(X)$ with either the LP or a W_p metric. Then:

$$\boxed{\underline{\dim}(X) \leq \underline{\mathrm{mo}}(\mathcal{M}(X)) \leq \overline{\mathrm{mo}}(\mathcal{M}(X)) \leq \overline{\dim}(X).}$$

In particular, if X has well-defined box dimension $\dim(X)$ then the metric order $\mathrm{mo}(\mathcal{M}(X))$ is well-def. and equals $\dim(X)$.

(*) Berger – B. – Peyre, after: Kulkarni–Zeitouni '95 ← LP

$\left.\begin{array}{l}\text{Bolley–Guillin–Villani '07}\\ \text{Kloeckner '15}\end{array}\right\} \leftarrow W_p$

Sketch of proof: (W_p)

$\boxed{\mathrm{mo}(\mathcal{M}(X)) \leq \dim(X)}$

Take a minimal ε-dense set $F = \{x_1, \dots, x_n\} \subseteq X$.
Take (appropriate) large integer $k = k(\varepsilon, n) \gg 1$.
Consider set of prob. measures of the form
$\sum_{i=1}^{n} p_i \delta_{x_i}$ where each $p_i \in \{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$, $\sum p_i = 1$

- Show that this set is $O(\varepsilon)$-dense and has card. $= e^{O(n)}$.

$\boxed{\mathrm{mo}(M(X)) \geq \dim(X)}$ Prune this set of measure so that it becomes $\Omega(\varepsilon)$-separated, still with card. $= e^{\Omega(n)}$.

Part 2:
Topological emergence

Now the dynamics ...

(X,d) compact

$f: X \to X$ continuous

We define sets of (Borel probability) measures:

f-invariant measures   ergodic

$$\mathcal{M}(X) \supseteq \underbrace{\mathcal{M}_f}_{\text{compact}} \supseteq \underbrace{\mathcal{M}_f^{erg}}_{\text{totally bounded, Borel}}$$

<u>Question:</u> How big are those sets?

Def. The topological emergence of $f: X \to X$ is the function

$$\mathcal{E}_{top}(f)(\varepsilon) := D_{\mathcal{M}_f^{erg}}(\varepsilon),$$

the covering number of $\mathcal{M}_f^{erg}$ (metrized either with the LP or a W_p metric).

Note: The actual def. uses a relative covering number, but the difference is not important.

The lower extreme:

- If f is uniquely ergodic then the topological emergence is as small as possible: $\mathcal{E}_{top}(f)(\varepsilon) \equiv 1$.

The upper extreme:

$$\mathcal{E}_{top}(f)(\varepsilon) = D_{\mathcal{M}_f^{erg}}(\varepsilon) \leq D_{\mathcal{M}(X)}(\varepsilon)$$

Suppose (X,d) has well-def. box dimension.
By a previous theorem, $\mathrm{mo}(\mathcal{M}(X)) = \dim(X)$,
that is, $D_{\mathcal{M}(X)}(\varepsilon)$ is roughly stretched exponential with exponent $\dim(X)$.
This is an upper bound for topological emergence.

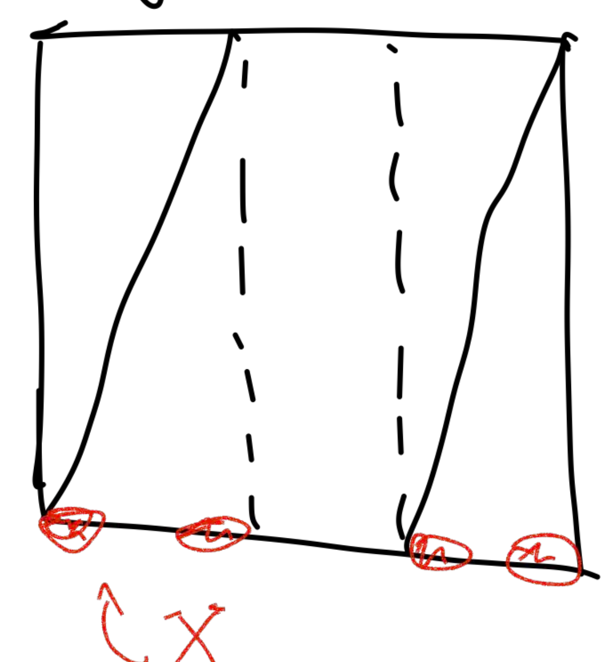
WE WILL SEE EXAMPLES WHERE THIS UPPER BOUND IS ESSENTIALLY ATTAINED.

Examples of high topological emergence

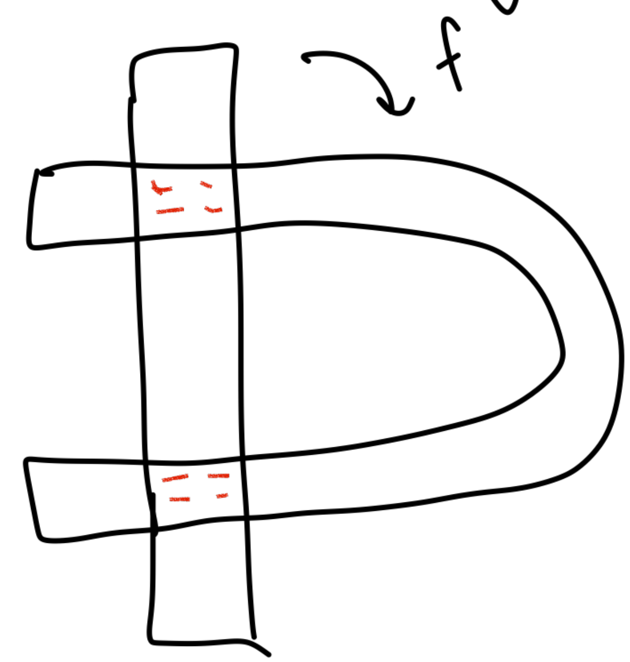
In the following examples, the dynamics $f : X \to X$ is such that

$$\boxed{\lim_{\varepsilon \to 0} \frac{\log \log \mathcal{E}_{top}(f)(\varepsilon)}{-\log \varepsilon} = \dim X}$$

Example 1: Any $C^{1+\alpha}$ topologically mixing conformal expanding repeller.



Example 2: Any horseshoe (locally maximal, topologically mixing, hyperbolic set) of a $C^{1+\alpha}$ area-preserving surface diffeomorphism.



Sketch of the proof of high top. emergence:

in the 1st example: $f: X \to X$ $C^{1+\alpha}$ conformal repeller

Lemma: To obtain the desired property, it is sufficient to find $\varepsilon_n \to 0$, and sets $\mathcal{F}_n$ of periodic orbits such that:

- any two distinct orbits in $\mathcal{F}_n$ are (uniformly) ε_n-apart from each other.
- $\# \mathcal{F}$ is roughly $\boxed{(1/\varepsilon_n)^d}$, where $d = \dim X$.

Very brief sketch of the proof of the Lemma:

- For each periodic orbit in $\mathcal{F}_{n}$, consider the corresponding ergodic probability measure.

- We form (carefully chosen) convex combinations of these measures, obtaining a <u>large</u> set of (non-ergodic) measures which is sufficiently separated (with respect to Wasserstein W_1 metric).

- By specification, we can approximate (as well as we want) these non-ergodic measures by ergodic measures (say, supported on periodic orbits of much higher period).

How to find the sets of periodic measures as in the Lemma?

Example: f = doubling map on $\mathbb{R}/\mathbb{Z}$.

Let $\boxed{\mathcal{F}_n := \{\text{periodic orbits of period } n\}}$

- Any two distinct such orbits are $\Omega(2^{-n})$ - apart
- There are $\Omega(2^n)$ such orbits

$\varepsilon_n = 2^{-n}$, $d = 1$, $\#\mathcal{F}_n$ roughly $(1/\varepsilon_n)^d$

DONE!

This approach doesn't work for general $f: X \to X$ because the minimal separation between two distinct orbits of period n can be too small.

<u>Solution to this difficulty:</u>

$\mu =$ prob. measure of maximal dimension $d = \dim X$

(equilibrium state of $\varphi = d \cdot \log|f'|$.)

$\lambda =$ Lyapunov exponent of μ.

$$\boxed{\mathcal{J}_n := \{\text{periodic orbits of period } n \text{ such that } |(f^n)'(x)| \text{ is roughly } e^{n\lambda}\}}$$

THEN:

- orbits in $\mathcal{F}_n$ are roughly $e^{-\lambda n}$-apart from each other.
- $\# \mathcal{F}_n$ is roughly $e^{h.n}$, $h = h_\mu(f)$.

$$\varepsilon_n = e^{-\lambda n} \Rightarrow (1/\varepsilon_n)^d = e^{\lambda d n} = e^{hn}$$

$$\lambda d = h$$

Done!

Problem:

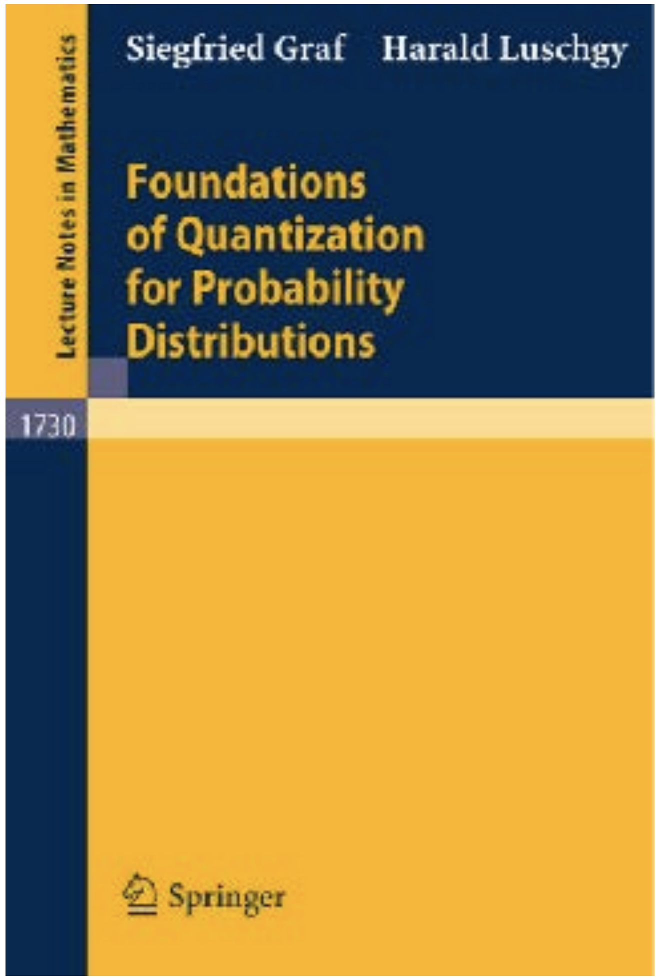
Compute topological emergence for:

→ non-conformal repellers

→ general hyperbolic sets

Is it maximal, or is it intermediate?

Part 3:
Quantization of measures



Let (Y,d) be a totally bounded Borel metric space.

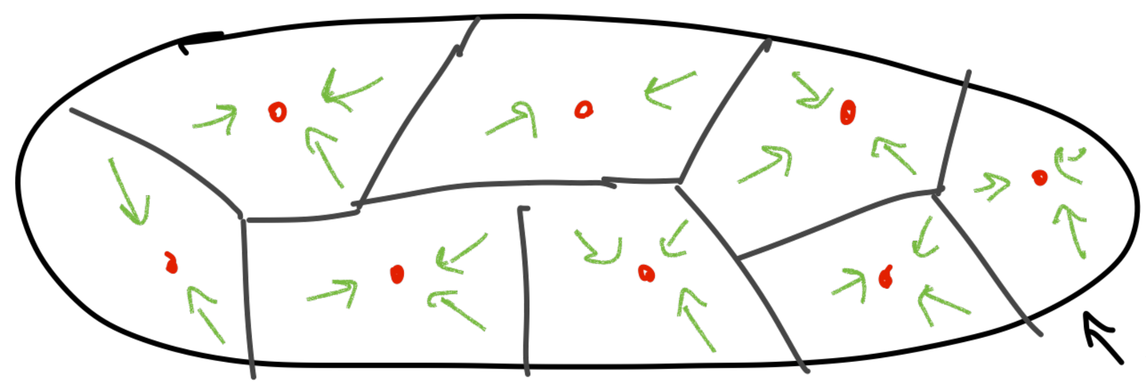
Metrize $\mathcal{M}(Y)$ using the Wasserstein W_q-metric. $1 \le q < \infty$.

Def. Given $\mu \in \mathcal{M}(Y)$ and $\varepsilon > 0$, the quantization number $Q_\mu(\varepsilon)$ is the least N such that there exists $\nu \in \mathcal{M}(Y)$ supported on N points, and ε-close to μ (w.r.t. W_q metric).

Equivalently:

$Q_\mu(\varepsilon)$ is the minimal cardinality N of a set $F \subseteq Y$ s.t.

$$\int_Y (d(x,F))^q \, d\mu(x) \leq \varepsilon^q .$$



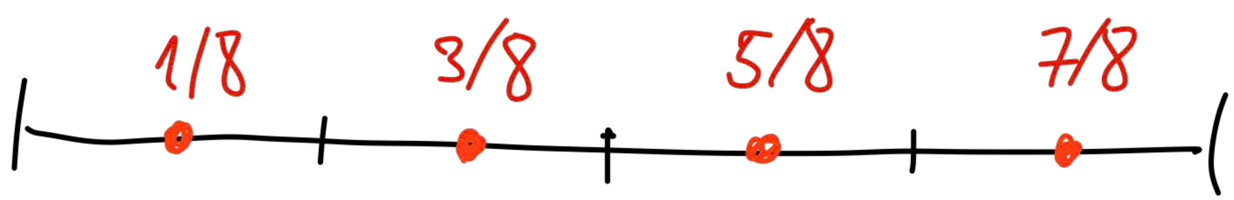
Consequence:

$\forall \mu \in M(Y)$, $\boxed{Q_\mu(\varepsilon) \leq D_Y(\varepsilon)}$ = covering number

(Indeed, if F is ε-dense then $\int \cdots \leq \varepsilon^q$.)

Example: $\begin{cases} Y = [0,1] & \text{with the usual metric} \\ \mu = \text{Lebesgue} \end{cases}$

The most efficient way of approximating Lebesgue by a measure ν supported on N points is to take ν equidistributed on $\left\{ \frac{1}{2N}, \frac{3}{2N}, \dots, \frac{2N-1}{2N} \right\}$



$(N=4)$

$$W_q(\mathrm{Leb}, \nu) = \Theta(1/N)$$

It follows that the quantization number is:

$$\boxed{Q_{\mathrm{Leb}}(\varepsilon) = \Theta(\varepsilon^{-1})}$$

(Informal) def. A measure $\mu \in \mathcal{M}(Y)$:

- has quantization dimension d if $Q_\mu(\varepsilon)$ is roughly $(1/\varepsilon)^d$.
  (it is always $\leq$ box dimension of Y)

- has quantization order θ if $Q_\mu(\varepsilon)$ is roughly $e^{(1/\varepsilon)^\theta}$.
  (it is always $\leq$ metric order of Y)

$$\begin{pmatrix}\text{quantization order}\\ \text{of any } \mu \in \mathcal{M}(Y)\end{pmatrix} \leq \begin{pmatrix}\text{metric order}\\ \text{of } (Y,d)\end{pmatrix}$$

<u>Theorem</u>. If (Y,d) has well-def. metric order θ then $\exists\ \mu \in \mathcal{M}(Y)$ whose quantization order is exactly θ.

Part 4:
Metric emergence

Let (X,d) be compact and $f: X \to X$ be continuous.

Let $\mu \in \mathcal{M}_f$ (a f-inv. Borel prob. meas. on X).

By the ergodic decomposition theorem,

μ is a convex combination of ergodic measures:

$$\exists\, \hat{\mu} \in \mathcal{M}(\mathcal{M}_f^{erg}) \text{ s.t. } \mu = \int_{\mathcal{M}_f^{erg}} \nu \, d\hat{\mu}(\nu).$$

Moreover, $\hat{\mu}$ is unique, and there is a bijection

$$\mathcal{M}_f \longleftrightarrow \mathcal{M}(\mathcal{M}_f^{erg})$$

$$\mu \longleftrightarrow \hat{\mu}$$

Metrize $\mathcal{M}(X)$ with the Wasserstein metric W_p and $\mathcal{M}(\mathcal{M}(X))$ with W_1.

Definition. The metric emergence of $\mu \in \mathcal{M}_f(X)$ is the quantization number of its ergodic decomposition $\hat{\mu}$ (considered as a measure on $\mathcal{M}(X)$):

$$\mathcal{E}_\mu(f)(\varepsilon) := Q_{\hat{\mu}}(\varepsilon).$$

Example (the lower extreme): If μ is ergodic then $\mathcal{E}_\mu(f)(\varepsilon) \equiv 1$.

Equivalent definition

(Berger, Proc. Steklov, 2017):

$\mathcal{E}_\mu(f)(\varepsilon)$ is the least number of ergodic measures $\mu_1, \mu_2, \dots, \mu_N$ such that:

$$\int \min_{1 \le i \le n} W_p\Big(\underbrace{\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}}_{\text{"empirical measure"}}, \mu_i \Big) \, d\mu(x) \le \varepsilon.$$

General fact: $\forall f : X \to X \quad \forall \mu \in \mathcal{M}_f$

$$\boxed{\underset{\text{of } f,\mu}{\text{Metric emergence}} \leq \underset{\text{of } f}{\text{Topological emergence}}}$$

Indeed:

$$\mathcal{E}_\mu(f)(\varepsilon) \underset{\substack{\uparrow \\ \text{def}}}{=} Q_{\hat{\mu}}(\varepsilon) \underset{\substack{\uparrow \\ \text{a previous} \\ \text{observation.}}}{\leq} D_{\mathcal{M}_f^{erg}}(\varepsilon) \underset{\substack{\uparrow \\ \text{def}}}{=} \mathcal{E}_{top}(f)(\varepsilon)$$

The upper bound is attained, by the following abstract result:

Theorem (Variational principle for emergence)

$\exists \mu \in \mathcal{M}_f$ s.t. $\mathcal{E}_\mu(f)(\varepsilon)$ and $\mathcal{E}_{top}(f)(\varepsilon)$ have the same <u>order</u>.

(where order of a function $\varphi(\varepsilon)$ is

$$\lim_{\varepsilon \to 0} \frac{\log \log \varphi(\varepsilon)}{-\log \varepsilon}$$

).

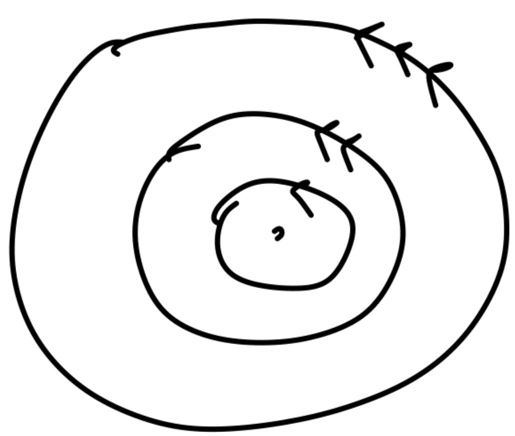
But the measure $\mu \in M_f$ in this theorem is highly artificial.

<u>Question</u>: Is there some natural class of examples?

Part 5:
Metric emergence of conservative dynamics

From now on, we will assume that f is a C^∞ area-preserving surface diffeo. and $\mu =$ area ("Lebesgue")

<u>Example:</u> A elliptic fixed point of "integrable" type



polar coordinates

$$(\theta, r) \longmapsto (\theta + \omega(r), r)$$

suppose $\omega' \neq 0$ (so most values of φ are irrational).

Then the ergodic decomposition of Lebesgue is "a Lebesgue on each circle".

(CONTINUES...)

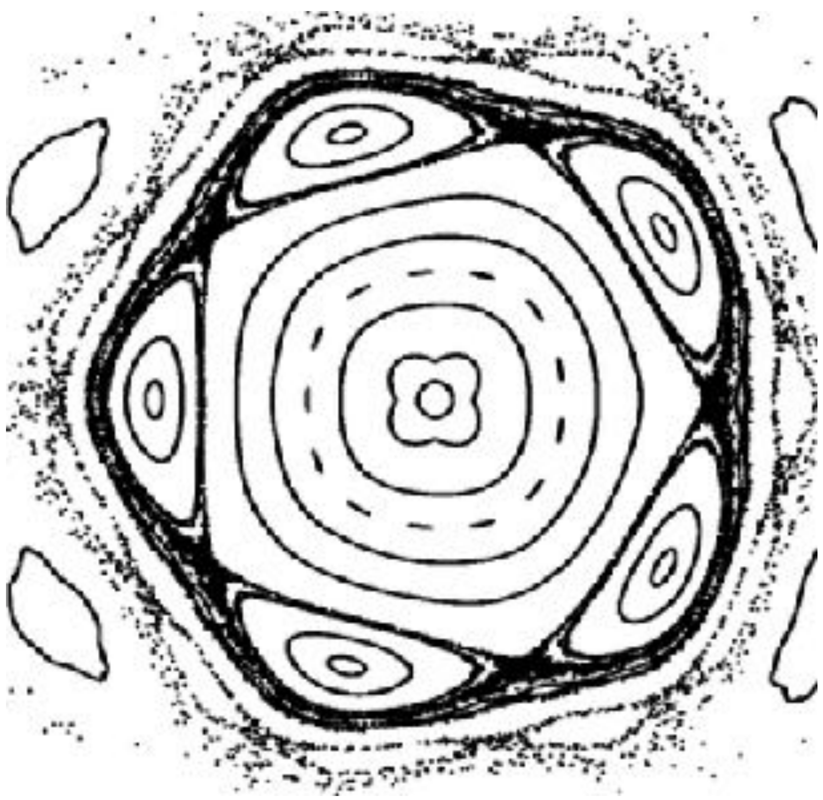
That is: $\quad \mu = Leb|_{disk} \Rightarrow \hat{\mu}$ "$=$" $Leb|_{[0,1]}$

And therefore

$$\mathcal{E}_{Leb}(f)(\varepsilon) = Q_{Leb|_{[0,1]}}(\varepsilon) \underset{\substack{\uparrow \\ \text{seen} \\ \text{before.}}}{=} \Theta(\varepsilon^{-1})$$

So f has "polynomial", actually "linear" metric emergence.

$\boxed{\text{KAM theorem. } g \ C^\infty\text{-perturbation of } f \Rightarrow \text{Most invariant circles persist (the "most" irrational ones)}}$



In particular, g has at least linear metric emergence.

QUESTION: Is it bigger?

(An invariant set of small (but positive) measure can have a very large emergence...)

On the other hand, the maximal emergence "allowed" by the (2-dimensional) ambient is roughly $e^{(1/\varepsilon)^2}$ (streched exponential with exponent 2).

<u>Theorem</u>. There exist a C^∞ area-preserving surface diffeo. with "maximal" metric emergence:

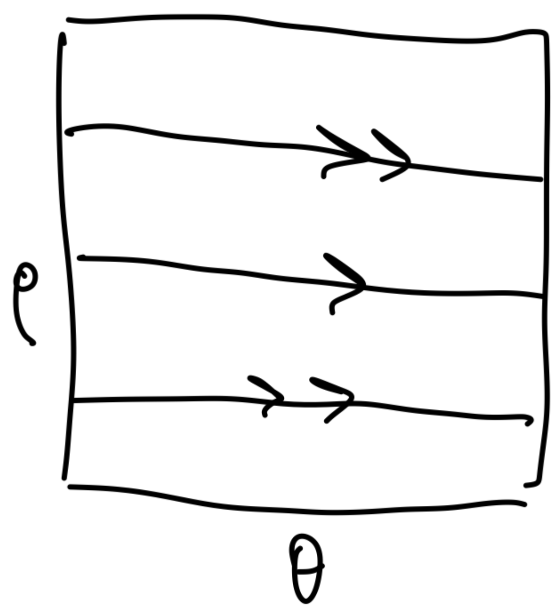
$$\liminf_{\varepsilon \to 0} \frac{\log \log \mathcal{E}_{Leb}(f)(\varepsilon)}{-\log \varepsilon} = 2.$$

Actually, f is the time 1 of an area preserving flow on the annulus $\mathbb{R}/\mathbb{Z} \times [0,1]$. (In particular, $h_{top}(f) = 0$).

Basic definitions:

$\mathbb{T} := \mathbb{R}/\mathbb{Z}$ circle $\qquad \mathbb{A} := \mathbb{T} \times [0,1]$ annulus

Horizontal flow with speed $\omega : [0,1] \to \mathbb{R}$

$R^t_\omega \circlearrowright$ 

speed at $(\rho, \theta) \in \mathbb{A}$
is $(\omega(\rho), 0)$

Proof of the theorem

First step: Constructing high emergence at a given scale $\varepsilon_* > 0$.

Key proposition. $\forall \varepsilon_* > 0 \ \exists h \in \mathrm{Diff}^\infty_{Leb}(\mathbb{A})$

$\forall \omega \in C^\infty([0,1], \mathbb{R})$ with $\omega' \neq 0$

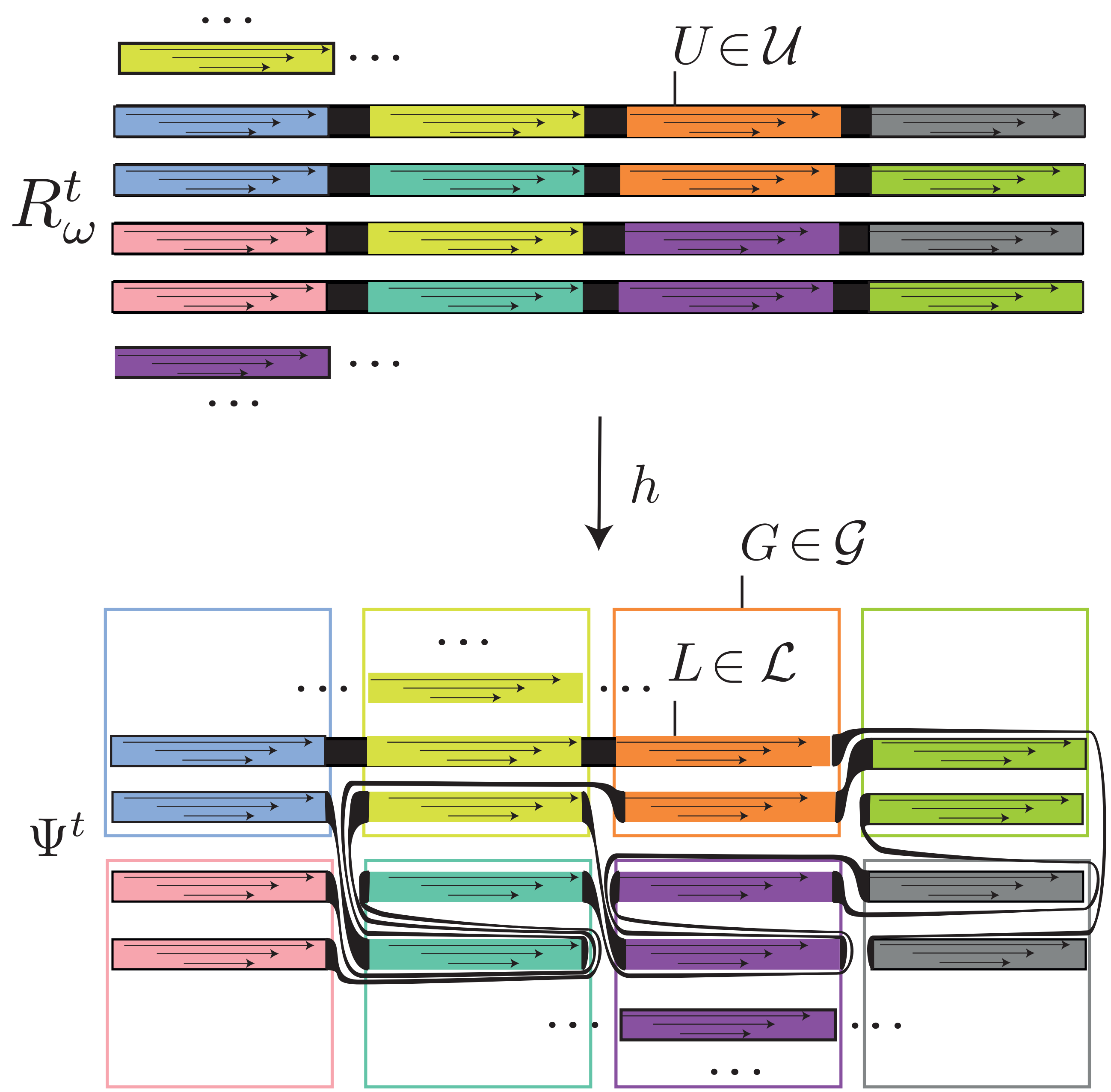
letting $\boxed{\Phi^t := h \circ R_\omega^t \circ h^{-1}}$ then $\forall t \neq 0$,

$$\mathcal{E}_{Leb}(\Phi^t)(\varepsilon_*) > e^{C\varepsilon_*^{-2}}.$$

(C = absolute constant.)

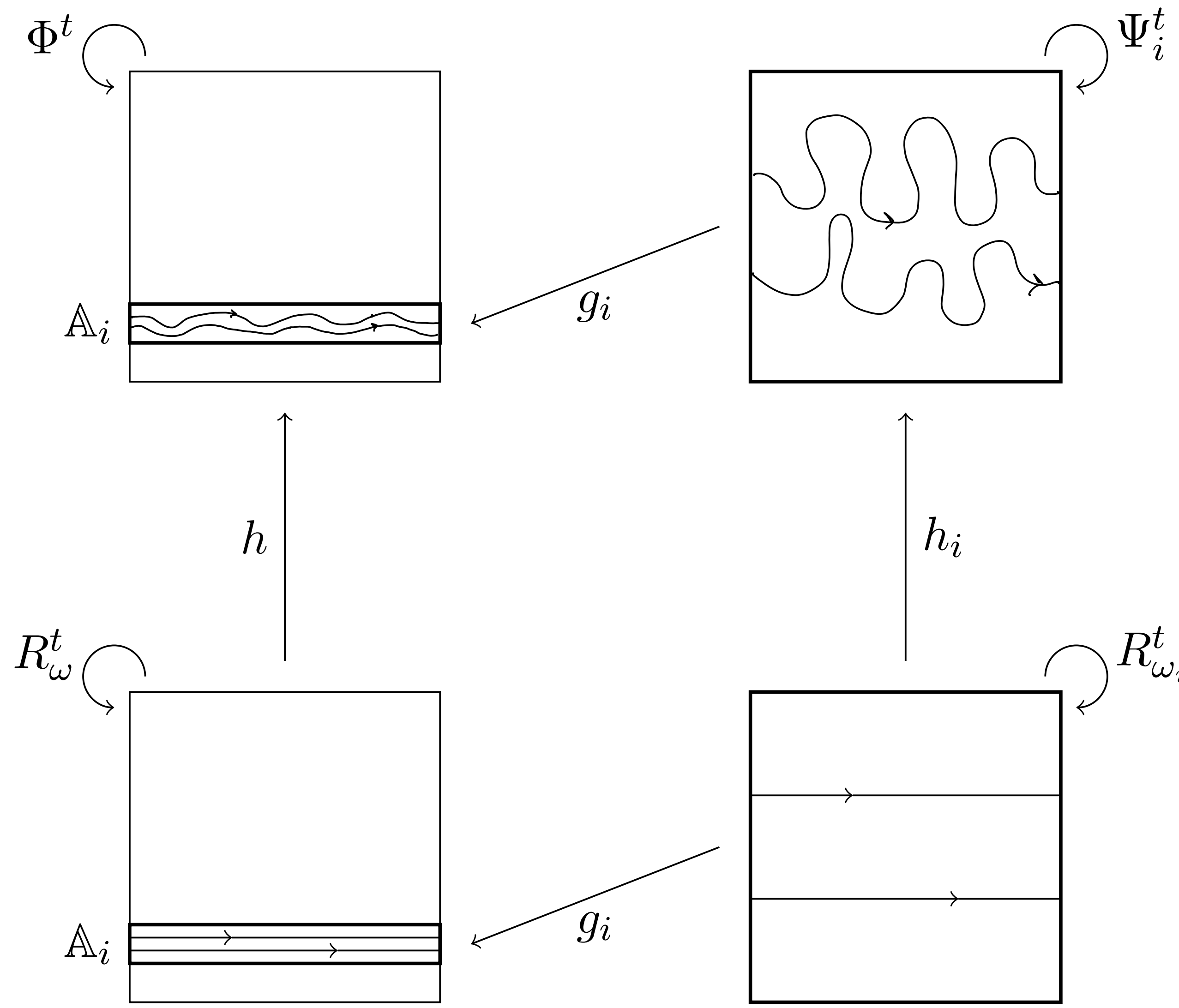
How to construct h?

The idea is to "separate" horizontal strips...



Combinatorial coloring argument to estimate emergence...

Second and last step of the proof of the theorem:
Insert ∞'ly many examples given by the Key Prop. in a single one:



Taking $\omega_i \to 0$ suff$^{\text{ly}}$ fast we obtain a C^∞ flow (Φ^t) with high emergence at every scale.

We have proved the existence of very high emergence among area-preserving diffeomorphisms.

How common is this behavior?

<u>Ergodicity</u> is the antithesis of high (metric) emergence, and ergodicity usually is a consequence of <u>hyperbolicity</u>.

Def. A (conservative) diffeo. is **weakly stable** if if all its periodic points are hyerbolic and the same is true for all (conservative) diffeos. nearby.

→ Uniformly hyp. (i.e. Anosov) diffeos. are weakly stable.

Conjecture: The converse is true.

A generic dichotomy:

Theorem. $\forall$ surface M $\exists$ residual (dense G_δ) $\mathcal{R} \subseteq \mathrm{Diff}^\infty_{Leb}(M)$ s.t. $\forall f \in \mathcal{R}$

EITHER f is weakly stable

OR $\displaystyle \limsup_{\varepsilon \to 0} \frac{\log \log \mathcal{E}_{Leb}(f)(\varepsilon)}{-\log \varepsilon} = 2.$

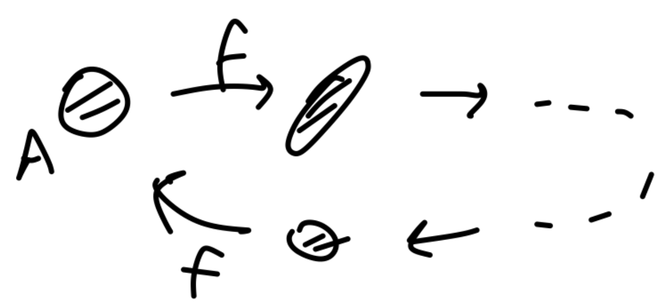
(maximal emergence)

Sketch of the proof:

(1) Not weakly stable $\Longrightarrow$ C^∞-perturb and create an elliptic periodic pt. $\Longrightarrow$

C^∞-perturb and create a periodic spot

(an open set A s.t. $f^n(A) = A$, $f^n|_A = id_A$



Goncheriko – Turaev – Shilnikov 2007

Gelfreich – Turaev 2010

(2) Insert our previous example of flow with high emergence inside the periodic spot.

(3) KAM theorem implies that high emergence persists under perturbations

(4) Baire argument concludes the proof.

Generic dynamics is not necessarily typical (from a probabilistic viewpoint).

CENTRAL PROBLEM: How big is the metric emergence of typical conservative dynamics?

Boundary of the "stochastic sea" of the Standard Map.