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MASTER THESIS

# Ergodic Optimization and Rotation Sets

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# Contents

# Chapter 1

# Introduction

Let $(X,T)$ be a topological dynamical system, that is, a compact metric space $X$ together with a continuous map $T : X \to X$. We denote by $\mathcal{M}_T$ be the set of $T$-invariant probability measures. Let $f : X \to \mathbb{R}$ be a continuous function, called a *performance function*. We denote the $n$-th Birkhoff sum of $f$ by:

$$f^{(n)} = f + f \circ T + ... + f \circ T^{n-1}.$$

The main purpose in *Ergodic Optimization* [Je2, Je4, B] is to understand the largest (or the least) value of Birkhoff averages of a performance function. More formally, the study of the quantity:

$$\beta(f) = \sup_{x \in R_f} \lim_{n\to\infty} \frac{f^{(n)}(x)}{n}, \tag{1.1}$$

where $R_f$ is the set of points $x \in X$ such that $\lim_{n\to\infty} \frac{f^{(n)}(x)}{n}$ exists, which has full measure for every measure $\mu \in \mathcal{M}_T$. The quantity (1.1) can be interpreted as the largest average value of $f$ along all the orbits of the dynamical system, that is:

$$\beta(f) = \sup_{\mu \in \mathcal{M}_T} \int f \, \mathrm{d}\mu. \tag{1.2}$$

Therefore, we are searching for the largest average of $f$ with respect all the ways of measuring the space, but keeping the dynamics of the space invariant. The quantity $\beta(f)$ is also called *the ergodic supremum*. A measure $\mu \in \mathcal{M}_T$ for which the supremum in (1.2) is attained is called a $f$-maximizing measure. Similarly, we define $\alpha(f) = -\beta(-f)$ to be *the ergodic infimum*, and analogously define $f$-minimizing measures. The main problem of ergodic optimization is to find a good description of maximizing measures, see the surveys [B, Je4] for full discussion.

A multidimensional generalization of this problem can be given as follows: fix an integer $d \geq 1$ and a continuous vector valued function $F : X \to \mathbb{R}^d$, we define its *rotation set* as:

$$R(F) = \left\{ \int F \, \mathrm{d}\mu : \mu \in \mathcal{M}_T \right\},$$

which is a non-empty compact and convex subset of $\mathbb{R}^d$. Note that in particular if $d = 1$, then $R(F) = [\alpha(F), \beta(F)]$. This generalization is called *Vectorial Ergodic Optimization* [B, section 2]. Now, we are ready to state the main result of this thesis:

**Theorem 1.1.** *Let $T : X \to X$ be a non-uniquely ergodic topological dynamical system with dense set of periodic measures. Then the map*

$$R : (C(X, \mathbb{R}^d), \|\cdot\|_\infty) \to (CB(\mathbb{R}^d), d_H)$$

*that associates to each potential $F$ its rotation set $R(F)$ is continuous, open, and surjective.*

Here, $C(X, \mathbb{R}^d)$ is endowed with the uniform norm, and $CB(\mathbb{R}^d)$ is the set of convex bodies of $\mathbb{R}^d$ endowed with the Hausdorff distance. As a consequence of our main result, we have:

**Corolary 1.2.** Let $T : X \to X$ be a non-uniquely ergodic topological dynamical system with dense set of periodic measures. Then there exists a residual subset $\mathcal{R}$ of $C(X, \mathbb{R}^d)$ such that $R(F)$ is strictly convex and has $C^1$ boundary for all $F \in \mathcal{R}$.

The purpose of this thesis is to expose and prove some remarkable results in Ergodic Optimization and to show newer results about the geometry of Rotation Sets.

*Organization of the Thesis:* The rest of this thesis is organized as follows. In Chapter 2 we prove the equivalence between optimizing Birkhoff averages and integral with respect $T$-invariant probability measures. Also, we prove the generic uniqueness of maximizing measures. In Chapter 3 we prove a result due to Jenkinson in the inverse direction of Ergodic Optimization, that is, given a ergodic measure, see if it is uniquely maximizing for some continuous performance potential. In Chapter 4 Manñé Lemma in the expanding case following a proof given by Jenkinson. Also, we prove the bilateral version proved by Bousch. In Chapter 5 we prove the main result of the thesis in the context of Rotation Sets.

# Chapter 2

# Generic Uniqueness of Maximizing Measures

Let $(X, T)$ be a topological dynamical system. Recall that given a continuous function $f : X \to \mathbb{R}$ we are interesed in the quantity

$$\beta(f) = \sup_{x \in R_f} \lim_{n\to\infty} \frac{1}{n} f^{(n)}(x), \tag{2.1}$$

where $R_f$ is the set of points for which $\lim_{n\to\infty} \frac{1}{n} f^{(n)}(x)$ exists. This quantity can be interpreted into an ergodic quantity, which is stated as follows:

**Proposition 2.1.** Let $f : X \to \mathbb{R}$ be a continuous function. Then:

$$\beta(f) = \sup_{x \in R_f} \lim_{n\to\infty} \frac{1}{n} f^{(n)}(x) = \sup_{x\in X} \overline{\lim_{n\to\infty}} \frac{1}{n} f^{(n)}(x) = \overline{\lim_{n\to\infty}} \sup_{x\in X} \frac{1}{n} f^{(n)}(x) = \sup_{\mu \in \mathcal{M}_T} \int f \, \mathrm{d}\mu. \tag{2.2}$$

Recall that a $f$-maximizing measure is a measure for which the supremum in (2.2) is attained. We denote by $\mathcal{M}_T^{\max}(f)$ the set of $f$-maximizing measures. Note that $\mathcal{M}_T^{\max}(f)$ is non-empty: if we take a sequence $(\mu_n)_{n\in\mathbb{N}}$ such that $\int f \, \mathrm{d}\mu_n \to \sup_{\mu\in\mathcal{M}_T} \int f \, \mathrm{d}\mu$, by weak-$*$ compactness there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ which weak $*$ converges to a measure $\mu_{\max} \in \mathcal{M}_T$. Then

$$\int f \, \mathrm{d}\mu_{\max} = \lim_{k\to\infty} \int f \, \mathrm{d}\mu_{n_k} = \sup_{\mu\in\mathcal{M}_T} \int f \, \mathrm{d}\mu. \tag{2.3}$$

Therefore the supremum in (2.2) is always attained by some measure in $\mathcal{M}_T$. Moreover, using the Ergodic Decomposition Theorem in a measure in $\mathcal{M}_T^{\max}(f)$, we can ensure the existence of ergodic $f$-maximizing measures.

*Proof of Proposition 2.1.* Let $\mu_{\max}$ be an ergodic $f$-maximizing measure. By the Birkhoff Ergodic Theorem there exists $x \in X$ such that:

$$\lim_{n\to\infty} \frac{1}{n} f^{(n)}(x) = \int f \,\mathrm{d}\mu_{\max} = \sup_{\mu\in\mathcal{M}_T} \int f \,\mathrm{d}\mu.$$

Therefore $\sup_{\mu\in\mathcal{M}_T} \int f \,\mathrm{d}\mu \leq \beta(f)$. Next, note that:

$$\beta(f) = \sup_{x\in R_f} \lim_{n\to\infty} \frac{1}{n} f^{(n)}(x) \leq \sup_{x\in X} \overline{\lim_{n\to\infty}} \frac{1}{n} f^{(n)}(x) \leq \overline{\lim_{n\to\infty}} \sup_{x\in X} \frac{1}{n} f^{(n)}(x).$$

Thus it suffices to prove the inequality

$$\overline{\lim_{n\to\infty}} \sup_{n\in\mathbb{N}} \frac{1}{n} f^{(n)}(x) \leq \sup_{\mu\in\mathcal{M}_T} \int f \,\mathrm{d}\mu.$$

Assuming the contrary, there exists a sequence $(x_{n_k})_{k\in\mathbb{N}}$ such that:

$$\lim_{k\to\infty} \frac{1}{n_k} f^{(n_k)}(x_{n_k}) > \sup_{\mu\in\mathcal{M}_T} \int f \,\mathrm{d}\mu.$$

Now, define the sequences of probability measures:

$$\mu_k = \frac{\delta_{x_{n_k}} + \delta_{T(x_{n_k})} + \ldots + \delta_{T^{n_k-1}(x_{n_k})}}{n_k}.$$

Passing to a subsequence, we may assume that $\mu_k$ converges to a probability measure $\nu$, which belongs to $\mathcal{M}_T$. Therefore

$$\int f \,\mathrm{d}\nu = \lim_{k\to\infty} \int f \,\mathrm{d}\mu_k = \lim_{k\to\infty} \frac{1}{n_k} f^{(n_k)}(x_{n_k}) > \sup_{\mu\in\mathcal{M}_T} \int f \,\mathrm{d}\mu,$$

which is a contradiction. ■

As we said in the introduction, the main problem is to obtain a good description of $f$-maximizing measures for $f \in C(X)$. It is natural to ask about the uniqueness of maximizing measures. We are going to prove the following classic fact, which states the generic uniqueness of maximizing measures:

**Theorem 2.2.** *([Je2, Theorem 2.4]) Let $V$ be a topological vector space densely and continuously embedded in C(X). Then, the set*

$$\{f \in V : \mathcal{M}_{\max}(f) \text{ is a singleton}\}, \tag{2.4}$$

*is a residual subset of $V$.*

Therefore, generic potentials in a space which is nicely embedded in $C(X)$ have a unique maximizing measure. The word *generic* is used to speak in a general topological sense, however, this result also holds in a probabilistic sense, which is called *prevalent*: for a complete proof and discussion see [Mo]. Furthermore, Contreras considered the case where $T$ is a hyperbolic dynamic and $f$ belong to a class of regular potentials, and he proved that the maximizing measures are generically supported on periodic orbits [Co]. Moreover, Huang, Lian, Ma, Xu and Zhang proved the same result but with a different proof. For, for specific formulation and proof see [H$^+$].
To prove Theorem 2.2 we follow [Je2, Mo]. The main idea is to prove that the set (2.4) is a countable intersection of continuity points of upper semi-continuous functions. For this purpose, we need the following lemma:

**Lemma 2.3.** *Let $g \in V$. Then, the map $L_g : V \to \mathbb{R}_{\geq 0}$ defined by:*

$$L_g(f) = diam \left\{ \int g \, d\mu : \mu \in \mathcal{M}_T^{\max}(f) \right\},$$

*is upper semi-continuous.*

*Proof.* Assume by contradiction that there exists $\varepsilon > 0$ and a sequence of functions $(f_n)_{n \in \mathbb{N}}$ converging to a function $f \in V$ such that:

$$\lim_{n \to \infty} L_g(f_n) \geq L_g(f) + \varepsilon.$$

By compactness of $M_{\max}(f_n)$ for each $n \in \mathbb{N}$ we can write $L_g(f_n)$ as

$$L_g(f_n) = \int g \, \mathrm{d}\mu_n^+ - \int g \, \mathrm{d}\mu_n^-.$$

Passing to a subsequence we can suppose without loss of generality that $\mu_n^\pm \to \mu^\pm \in \mathcal{M}_T$, since $\mathcal{M}_T$ is compact and metrizable. Note that for every $n \in \mathbb{N}$:

$$\int f_n \, \mathrm{d}\mu_n^\pm \geq \int f_n \, \mathrm{d}\mu \; \forall \mu \in \mathcal{M}_T,$$

and passing to the limit when $n \to \infty$ we get that $\int f \, \mathrm{d}\mu^\pm \geq \int f \, \mathrm{d}\mu$ for all $\mu \in \mathcal{M}_T$, which implies that $\mu^\pm \in \mathcal{M}_T^{\max}(f)$. Therefore:

$$L_g(f) \geq \int g \, \mathrm{d}\mu^+ - \int g \, \mathrm{d}\mu^- = \lim_{n \to \infty} \int g \, \mathrm{d}\mu_n^+ - \int g \, \mathrm{d}\mu_n^- = \lim_{n \to \infty} L_g(f_n) \geq L_g(f) + \varepsilon,$$

which is a contradiction. We conclude that $L_g$ is upper semi continuous for all $g \in V$. ■

We need another result which characterises the set of continuity points of $L_g$, which is stated as follows:

**Lemma 2.4.** *([Je2, Lemma 3.4]) Let $f, g \in V$. Then $L_g$ is continuous at $f \in V$ if and only if $L_g(f) = 0$.*

*Proof.* Let $f \in V$ be a continuity point of $L_g$. Take a sequence $\varepsilon_n \to 0$ and write $L_g$ as:

$$L_g(f + \varepsilon_n g) = \int g \,\mathrm{d}\mu_n^+ - \int g \,\mathrm{d}\mu_n^-,$$

for each $n \in \mathbb{N}$. Also, suppose that the sequences $\mu_n^\pm$ weak-* covnverge to $\mu^\pm \in \mathcal{M}_T$. Doing the same thing as in the proof of Lemma 2.3, see that $\mu^\pm \in \mathcal{M}_{\max}(f)$. Now, note that for every $\mu \in \mathcal{M}_{\max}(f)$:

$$\int f + \varepsilon g \,\mathrm{d}\mu_n^\pm \geq \int f + \varepsilon g \,\mathrm{d}\mu$$

$$0 \geq \frac{\int f \,\mathrm{d}\mu_n^\pm - \int f \,\mathrm{d}\mu}{\varepsilon_n} \geq \int g \,\mathrm{d}\mu - \int g \,\mathrm{d}\mu^\pm, \tag{2.5}$$

therefore applying the limit in (2.5) when $n \to \infty$ we have that $\int g \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu^\pm$ for every $\mu \in \mathcal{M}_{\max}(f)$. Thus $L_g(f) = 0$.
For the opposite direction, it suffices to note that by Lemma 2.3 $L_g$ is upper semicontinuous and non-negative, therefore the functions $f \in V$ for which $L_g = 0$ are continuity points of $L_g$. ∎

Now we are ready to prove Theorem 2.2.

*Proof.* Let $\{g_1, g_2, ...\}$ be a countable dense subset of $V$. Note that $f \in C(X)$ satisfies that $\mathcal{M}_T^{\max}(f)$ is a singleton if and only if $L_g(f) = 0$ for all $g \in V$. This is also equivalent to $L_{g_n}(f) = 0$ for every $n \in \mathbb{N}$. Therefore by Lemma 2.4 we have that

$$\{f \in V : \mathcal{M}_{\max}(f) \text{is a singleton}\} = \bigcap_{n \in \mathbb{N}} \{f \in V : L_g \text{ is continuous at } f\},$$

which is residual since $L_{g_n}$ is upper semi-continuous by Lemma 2.3 for all $n \in \mathbb{N}$. ∎

# Chapter 3

# Jenkinson's Theorem: Every ergodic measure is uniquely maximizing

## 3.1 Introduction

Let $(X, T)$ be a topological dynamical system. Note that if there exists an unique maximizing measure for a given potential $f \in C(X)$, it has to be ergodic. Thus, it is natural to ask a inverse question: for which ergodic measures on $\mu \in M_T$ there exists a continuous potential $f \in C(X)$ such that $\mathcal{M}_T^{\max}(f) = \{\mu\}$. This question was answered by Jenkinson [Je3]:

**Theorem 3.1.** *([Je3, Theorem 1]) Let $X$ a compact metric space together with a continuous map $T : X \to X$. Then for every ergodic measure $\mu \in \mathcal{M}_T^{erg}$ there exists a continuous potential $f \in C(X)$ such that $\mathcal{M}_T^{\max}(f) = \{\mu\}$.*

The aim of this chapter is to show the proof given by Jenkinson. The proof of this theorem breaks into two steps. The first one is to use the simplex structure of $\mathcal{M}_T$ in order to verify that every extreme point $\mu \in \mathcal{M}_T$ is exposed. The second step is to prove a sort of Riesz Representation Theorem for affine functionals defined on $\mathcal{M}_T$.

## 3.2 $\mathcal{M}_T$ has the structure of a simplex

Let $\mathcal{M}(X)$ be the space of signed borel measures, which is a topological vector space equipped with the weak-$*$ topology. Moreover, this topology is locally convex and metrizable. Also, $\mathcal{M}(X)$ is a Riesz space: it is an partially ordered vector space equipped with the order induced by the cone $\mathcal{M}_+(X)$ of all positive Borel measures on $X$. In addition, it is a lattice with operations:

$$(\mu \vee \nu)(A) = \sup\{\mu(B) + \nu(A \setminus B) : B \in \mathcal{B}, B \subset A\},$$

$$(\mu \wedge \nu)(A) = \inf\{\mu(B) + \nu(A \setminus B) : B \in \mathcal{B}, B \subset A\},$$

that is, $\mu \vee \nu$ and $\mu \wedge \nu$ are the infimum and supremum of the two elements $\mu$ and $\nu$, respectively. Note that $\mu^+ = \mu \vee 0$ and $\mu^- = \mu \wedge 0$, where $\mu^+$ and $\mu^-$ are the positive measures given in the classical Jordan decomposition. We denote by $\mathcal{M}_T^\pm$ the set of signed $T$-invariant measures. We are going to prove the following

**Theorem 3.2.** *The space $\mathcal{M}_T$ is a compact metrizable simplex.*

*Proof.* For the metrizability and compactness, see [OV, Chapter 2]. In order to prove that $\mathcal{M}_T$ is a simplex, note that $\mathcal{M}_T$ is contained in a hyperplane which does not contains the origin, and $\mathcal{M}_T$ is a basis for the cone of positive $T$-invariant measures $\mathcal{M}_T^+$ which satisfies $\mathcal{M}_T^+ - \mathcal{M}_T^+ = \mathcal{M}_T^\pm$. It suffices to prove that $\mathcal{M}_T^\pm$ is a sublattice of $\mathcal{M}(X)$ with respect to the order induced by the cone $\mathcal{M}^+(X)$ with operations $\wedge$ and $\vee$. In other words, we have to prove that if $\mu, \nu \in \mathcal{M}_T^\pm$ then $\mu \wedge \nu$ and $\mu \vee \nu$ belong to $\mathcal{M}_T^\pm$. It is not difficult to prove that for arbitrary $\mu, \nu \in \mathcal{M}_T^\pm$ we have that:

$$\mu + \nu = \mu \vee \nu + \mu \wedge \nu,$$

$$\mu = (\mu - \nu)^+ + \mu \wedge \nu.$$

Therefore it is enough to prove that $\mu^+ \in \mathcal{M}_T^\pm$ for all $\mu \in \mathcal{M}_T^\pm$. For this purpose let $E \in \mathcal{B}$, thus

$$\mu^+(T^{-1}E) - \mu^-(T^{-1}E) = \mu(T^{-1}E) = \mu(E) = \mu^+(E) - \mu^-(E),$$

which implies that $\mu^+(E) \leq \mu^+(T^{-1}E)$ for all $E \in \mathcal{B}$. Replacing $E$ by $E^c$ we obtain the reverse inequality, impliying that $\mu^+ \in \mathcal{M}_T^+$. ■

The principal property that Jenkinson uses in the proof of Theorem 3.1 is that every extreme point in a simplex is exposed, that is, there exists an affine functional defined on $\mathcal{M}_T$ which separates the extreme point with the rest of the space

**Theorem 3.3.** *A point $\mu \in \mathcal{M}_T$ is extreme if and only it is exposed.*

In general every exposed point in an arbitrary convex set is extremal, but the converse is not always true. For example, consider the case showed in Figure 3.1.

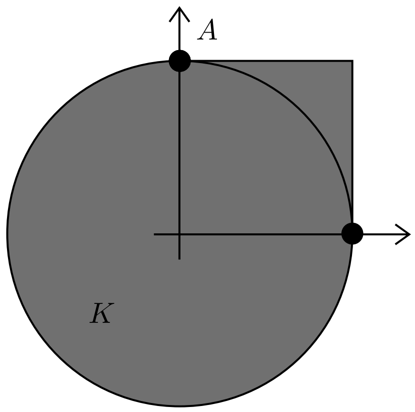



FIGURE 3.1: A convex set $K$ whose extreme point $A$ is not exposed.

A key fact in the proof of Theorem 3.3 is the following

**Theorem 3.4.** *([E]) Let $K$ be a simplex and let $f : K \to [-\infty, +\infty)$ be an upper semicontinuous convex function and $g : K \to (-\infty, +\infty]$ a lower semicontinuous concave function. If $f \leq g$, then there exists an affine functional $h : K \to \mathbb{R}$ such that $f \leq h \leq g$.*

For completeness we give the proof given in [Je3].

*Proof.* We prove that every extremal point is exposed. Let $\nu \in \mathcal{M}_T$ be a extremal point. We claim that for each measure $\mu \in \mathcal{M}_T$ there exists an affine functional $\ell_\mu : \mathcal{M}_T \to \mathbb{R}_{\geq 0}$ such that $\ell_\mu(\mu) > 0$ and $\ell_\mu(\nu) = 0$. To prove this claim, by the Hahn-Banach theorem, for each $\mu \in \mathcal{M}_T$ there exists an affine functional $\lambda_\mu : \mathcal{M}_T \to \mathbb{R}$ such that $\lambda_\mu(\nu) = 0$ and $\lambda_\mu(\mu) > 0$. Define $\eta_\mu, \xi_\mu : \mathcal{M}_T \to \mathbb{R}$ by:

$$\eta_\mu(m) = \max\{0, \lambda(m)\},$$

$$\xi_\mu(m) = \begin{cases} 0 & \text{if } m = \nu \\ \max|\lambda| & \text{if } m \neq \nu \end{cases}.$$

Note that $\eta_\mu \leq \xi_\mu$ and both satisfies the conditions of Theorem 3.4, thus there exists an affine linear functional $\ell_\mu : \mathcal{M}_T \to \mathbb{R}_{\geq 0}$ such that $\eta_\mu \leq \ell_\mu \leq \xi_\mu$. Moreover, we have:

$$\ell_\mu(\mu) \geq \eta_\mu(\mu) > 0$$

$$0 \leq \ell_\nu \leq \chi_\mu(\nu) = 0$$

and thus we proved the claim.

Next, we claim that given a closed set $\mathcal{N}$ not containing $\nu$, there exists an affine functional $\ell_{\mathcal{N}} : \mathcal{M}_T \to \mathbb{R}_{\geq 0}$ such that $\ell_{\mathcal{N}}(\nu) = 0$ and $\ell_{\mathbb{N}} \subset \mathbb{R}_{>0}$. For this, first take for each

$\mu \in \mathcal{N}$ the open set:

$$\mathcal{N}_\mu = \{\eta \in \mathcal{M}_T : \ell_\mu(\eta) > 0\}$$

Second, note that $\{\mathcal{N}_\mu\}_{\mu\in\mathcal{N}}$ forms an open cover of $\mathcal{N}$ which is compact, therefore there exists a finite sub cover $\{\mathcal{N}_{\mu_i}\}_{i=1}^n$. To prove the claim, just consider $\ell_{\mathcal{N}} = \sum_{i=1}^n \ell_{\mu_i}$.
For the last part, note that $\mathcal{M}_T \setminus \{\nu\} = \bigcup_{i=1}^\infty \mathcal{N}_i$, where the sets $\mathcal{N}_i$ are closed. Define

$$\ell = \sum_{i=1}^\infty \frac{\ell_{\mathcal{N}_i}}{k_i 2^i},$$

where $k_i = \max_{\mu\in\mathcal{M}_T} |\ell_{\mathcal{N}_i}(\mu)|$. This $\ell$ proves that $\nu$ is exposed. ■

It is a well known fact that the extremal points of $\mathcal{M}_T$ are exactly the ergodic measures, but for completeness we give a proof.

**Theorem 3.5.** *([W, Theorem 6.10 (iii)]) A measure $\mu \in \mathcal{M}_T$ is an extreme point of $\mathcal{M}_T$ if and only if $\mu$ is ergodic.*

*Proof.* Let $\mu \in \mathcal{M}_T$ be a extreme point of $\mathcal{M}_T$. Suppose that $\mu$ is not ergodic, i.e., there exists a Borel set $E$ such that $T^{-1}(E) = E$ and $\mu(E) \in (0,1)$. Define the conditional probability measures $\mu_1 \neq \mu_2$ as

$$\mu_1 = \mu(B|E) \text{ and } \mu_2(B) = \mu(B|E^c).$$

Since $E$ is invariant, we have that $\mu_1, \mu_2 \in \mathcal{M}_T$ and

$$\mu(E)\mu_1 + (1-\mu(E))\mu_2 = \mu.$$

Therefore $\mu$ is not a extreme point in $\mathcal{M}_T$, a contradiction.
Suppose now that $\mu$ is ergodic and

$$\mu = p\mu_1 + (1-p)\mu_2,$$

for $p \in (0,1)$ and $\mu_1, \mu_2 \in \mathcal{M}_T$. It is clear that $\mu_1 << \mu$, hence by Radon-Niykodym theorem we have:

$$\mu_1(E) = \int_E \rho(x)\,\mathrm{d}\mu(x),$$

for some non-negative $\rho \in L^1(\mu)$. We claim that $\rho \equiv 1$ $\mu$-almost everywhere. Let

$$E = \{x \in X : \rho(x) < 1\}.$$

Then by the invariance of $\mu_1$:

$$\int_{E\setminus T^{-1}(E)} \rho(x)\,\mathrm{d}\mu(x) = \int_{T^{-1}(E)\setminus E} \rho(x)\,\mathrm{d}\mu(x).$$

If $\mu(E \setminus T^{-1}(E)) > 0$ then:

$$\mu(T^{-1}(E) \setminus (E)) \leq \int_{T^{-1}(E)\setminus E} \rho(x)\,\mathrm{d}\mu(x) = \int_{E\setminus T^{-1}(E)} \rho(x)\,\mathrm{d}\mu(x) < \mu(E \setminus T^{-1}(E)),$$

which contradicts the invariance of $\mu$. Therefore $\mu(E \setminus T^{-1}(E)) = \mu(T^{-1}(E) \setminus E) = 0$, which is equivalent to $\mu(E\Delta T^{-1}(E)) = 0$, and by ergodicity $\mu(E) = 0$ or $\mu(E) = 1$.
If $\mu(E) = 1$, we have that

$$1 = \mu_1(X) = \int \rho(x)\,\mathrm{d}\mu(x) = \int_E \rho(x)\,\mathrm{d}\mu(x) < 1,$$

which is a contradiction. Thus $\mu(E) = 0$ impliying that $\rho \leq 1$ $\mu$-almost everywhere. Finally since $\mu_1$ is a probability and

$$1 = \mu_1(X) = \int \rho(x)\,\mathrm{d}\mu(x),$$

we have that $\rho \equiv 1$ $\mu$-almost everywhere, and this implies that $\mu = \mu_1$. Consequentely $\mu$ is a extreme point of $\mathcal{M}_T$. ■

## 3.3 A representation theorem

Now that we know that each ergodic measure is exposed, the next step is to find which form can take affine functionals of $\mathcal{M}_T$. The aim of this section is to prove the following representation theorem:

**Theorem 3.6.** *Suppose that $\ell : \mathcal{M}_T \to \mathbb{R}$ is weak-$*$ continuous and affine. Then, there exists $f \in C(X)$ such that:*

$$\ell(\mu) = \int f\,d\mu \qquad \forall \mu \in \mathcal{M}_T.$$

For this we need to state some terminology. Let $B_T$ be the closure in $C(X)$ of the subspace genrated by the set $\{f - f \circ T : f \in C(X)\}$. We denote by $\langle f, \mu \rangle$ the duality between $C(X)$ and the space of signed Borel masures on $X$. We have the following characterization of $B_T$.

**Lemma 3.7.** $B_T = \{h \in C(X) : \langle h, \mu \rangle = 0\ \forall \mu \in \mathcal{M}_T\}$

*Proof.* We note that

$$\{\mu \in \mathcal{M}(X) : \langle h, \mu \rangle = 0 \, \forall h \in B_T\} = \mathcal{M}_T^{\pm}.$$

This implies by [AB, Theorem 9.16] that the topological dual of $(\mathcal{M}_T^{\pm}, w^*)$ is $C(X)/B_T$. On the other hand, the topological dual of $(\mathcal{M}_T^{\pm}, w^*)$ is $C(X)/\mathrm{Ann}(\mathcal{M}_T^{\pm})$. Combining the two expressions for the topological dual of $(\mathcal{M}_T^{\pm}, w^*)$ yields the desired result. ■

The duality of the pair $(C(X)/B_T, \mathcal{M}_T^{\pm})$ will also be denoted by $\langle f + B_T, \mu \rangle$, which is clearly well defined. Now we prove Theorem 3.6.

*Proof.* We are going to extend $\ell : \mathcal{M}_T \to \mathbb{R}$ to an affine weak-* continuous linear functional $\widetilde{\ell} : \mathcal{M}_T^{\pm} \to \mathbb{R}$.

**First Step.** We extend $\ell : \mathcal{M}_T \to \mathbb{R}$ to an affine weak-* continuous linear functional defined on the cone generated by $\mathcal{M}_T$. The cone $\mathcal{M}_T^+$ of positive $T$-invariant measures can be written as:

$$\mathcal{M}_T^+ = \{c\mu : c \geq 0 \text{and} \mu \in \mathcal{M}_T\}.$$

So define $\ell_1 : \mathcal{M}_T^+ \to \mathbb{R}$ with the formula

$$\ell_1(\nu) = c\ell(\mu) \, \forall \mu \in \mathcal{M}_T^+ \setminus \{0\}, \text{ and } \ell_1(0) = 0 \,,$$

where $\nu = c\mu$ is the unique representation mentioned above. Note that $\ell_1$ is additive. We claim that $\ell_1$ is weak * continuous. The continuity at any nonzero measure follows from the continuity of $\ell$. To prove the continuity at $0 \in \mathcal{M}_T^+$, let $\nu_\alpha$ be a net in $\mathcal{M}_T^+$ such that $\nu_\alpha \to 0$. Write $\nu_\alpha = c_\alpha \mu_\alpha$ where $\mu_\alpha \in \mathcal{M}_T$. Next, see that

$$c_\alpha = c_\alpha \langle 1, \mu_\alpha \rangle = \langle 1, \nu_\alpha \rangle \to 0.$$

In addition compactness of $\mathcal{M}_T$ implies that $\{\ell(\nu_\alpha)\}$ is bounded independentely of $\alpha$. Therefore

$$\ell_1(\nu_\alpha) = c_\alpha \ell(\mu_\alpha) \to 0.$$

Consequentely $\ell_1$ is an affine weak * continuous functional.

**Second Step.** We extend $\ell_1$ to a weak * continuous linear functional $\widetilde{\ell} : \mathcal{M}_T^{\pm} \to \mathbb{R}$ defined by:

$$\widetilde{\ell}(\mu) = \ell_1(\mu^+) - \ell_1(\mu^-),$$

where $\mu^+$ and $\mu^-$ where defined in the previous section. The additivity of $\mu^+$, $\mu^-$ and $\ell_1$ implies that $\ell_1$ is linear. The main problem is the weak-* continuity, since the Jordan decomposition is not always continuous.

Since $\mathcal{M}_T^{\pm}$ is the topological dual of $C(X)/B_T$ it suffices to prove that the restriction of $\ell$ to the unit ball in $(\mathcal{M}_T^{\pm}, ||\cdot||)$ is continuous, where $||\cdot||$ is the dual norm. Note that $||\cdot||$ coincides with the total variation norm of measures in $\mathcal{M}_T^{\pm}$. The linearity of $\ell_1$ implies that $\widetilde{\ell}$ is continuous if and only if $B \cap \ker \widetilde{\ell}$ is weak *-closed.
Take $\{\mu_n\}_{n\in\mathbb{N}} \subset B \cap \ker \widetilde{\ell}$ such that $\mu_n \to \mu$, so it suffices to prove that $\mu \in \ker \widetilde{\ell}$. By the compactness of $B$, there exists a subsequence $\mu_{n_k}$ of $\mu_n$ such that $\mu_{n_k}^+ \to \nu_1$ and $\mu_{n_k}^- \to \nu_2$. This implies that

$$\mu = v_1 - v_2,$$

and therefore

$$\begin{aligned}\widetilde{\ell}(\mu) &= \widetilde{\ell}(\nu_1 - \nu_2)\\ &= \widetilde{\ell}(\nu_1) - \widetilde{\ell}(\nu_2)\\ &= \ell_1(\nu_1) - \ell_1(\nu_2)\\ &= \lim_{k\to\infty} \ell_1(\mu_{n_k}^+) - \ell_1(\mu_{n_k}^-)\\ &= \lim_{k\to\infty} \widetilde{\ell}(\mu_{n_k}) = 0.\end{aligned}$$

Thus $\mu \in \ker(\widetilde{\ell})$. We conclude that $\widetilde{\ell}$ is weak * continuous.
**Final step.** We conclude that $\widetilde{\ell}$ is a linear weak-* continuous extension of $\ell$, and since the topological dual of $(\mathcal{M}_T^{\pm}, w^*)$ is $C(X)/B_T$, there exists $f + B_T \in C(X)/B_T$ such that:

$$\widetilde{\ell}(\mu) = \langle f + B(X), \mu \rangle \quad \forall \mu \in \mathcal{M}_T^{\pm}.$$

In particular $\ell(\mu) = \int f \, \mathrm{d}\mu \, \forall \mu \in \mathcal{M}_T$. ■

## 3.4 Proof of the main theorem

Now we are ready to collect the results of the previous sections and prove the main theorem of this chapter.

*Proof.* Let $\mu \in \mathcal{M}_T$ be an ergodic measure. Since $\mu$ is an extreme point, by Theorem 3.3 we have that $\mu$ is exposed, that is, there exists a weak * continuous affine functional $\ell : \mathcal{M}_T \to \mathbb{R}$ such that

$$\ell(\nu) < 0 \text{ for all } \nu \in \mathcal{M}_T \setminus \{\mu\} \text{ and } \ell(\mu) = 0.$$

By Theorem 3.6, there exists a function $f \in C(X)$ such that $\ell(\nu) = \langle f, \nu \rangle$ for all $\nu \in \mathcal{M}_T$. Therefore we have that:

$$\sup_{\nu \in \mathcal{M}_T} \int f \, \mathrm{d}\nu = 0$$

and the supremum is attained at only $\nu = \mu$, as we wanted to prove. ■

# Chapter 4

# Two Mañé Lemmas

## 4.1 Introduction

There is a useful tool for the understanding of maximizing measures, namely *Mañé Lemma* [Sa, Bo1, CG, Bo2] or *Revelation lemma* [Je4]. For the purpose of the following chapter, we will use the same terminology of [Je4]. We will define the concept of *revealed function*:

**Definition 4.1.** Let $f : X \to \mathbb{R}$ be continuous. We say that $f$ is revealed if $f^{-1}(\max f)$ contains a compact forward $T$-invariant [1] set.

For revealed functions we have the following proposition, which helps the understanding of the ergodic maximum and the optimal measures in which the ergodic supremum is attained.

**Proposition 4.2.** Let $f : X \to \mathbb{R}$ be a revealed function. Then, the following hold:

1. $\beta(f) = \max f$;
2. $\mathcal{M}_T^{max}(f) = \{\mu \in \mathcal{M}_T : supp(\mu) \subset f^{-1}(\max f)\}$.

*Proof.* To prove (1), notice that for every $\mu \in \mathcal{M}_T$ we have $\int f d\mu \leq \max f$, so taking supremum over all those measures we get $\beta(f) \leq \max f$. Now since $f$ is revealed, there exists a compact $T$-invariant set $K$, so we consider the dynamic $T|_K : K \to K$. By Krylov-Bogolyubov's Theorem (see for example [OV, Chapter 2]), there exists $\mu_K \in \mathcal{M}_{T|_K}$. Now, for every Borel set $A \subset X$ define $\nu(A) := \mu_K(A \cap K)$. Using that $K$ is

[1] We say that a non-empty subset $A \subset X$ is forward $T$-invariant if $T(A) \subset A$.

$T$-invariant, it is not hard to show that $\nu \in \mathcal{M}_T$ and that $\nu(K) = 1$. Since $f \equiv \max f$ on $K$,

$$\max f = \nu(K) \max f = \int f d\nu \leq \beta(f).$$

In order to prove (2), let $\mu \in \mathcal{M}_T$ such that $supp(\mu) \subset f^{-1}(\max f)$. Then, by (2)

$$\int f d\mu = \int_{f^{-1}(\max f)} f d\mu = \max f = \beta(f),$$

and thus $\mu \in \mathcal{M}_T^{max}$. Now assume that $\mu \in \mathcal{M}_T^{max}$, by (1) we get

$$\int \underbrace{(\max f - f)}_{\geq 0} d\mu = 0,$$

which implies that $supp(\mu) \subset \{x \in X : f(x) = \max f\} = f^{-1}(\max f)$, and this concludes the proof of (2). ∎

Proposition 4.2 tells us that for revealed functions, we have a characterization of the maximizing measures. However, in general the functions are not revealed. For this purpose, given a function we try to find revealed functions that are in some sense equivalent to the original one.

**Definition 4.3.** Let $f : X \to \mathbb{R}$ be continuos. We say that $f$ has a weak-revelation $\psi \in C(X)$ if $\psi$ is a weak-coboundary [2] and $f + \psi$ is a revealed function.

A natural choice of $\psi$ is what is called a *continuous coboundary*, i.e. a function of the form $\varphi - \varphi \circ T$, where $\varphi \in C(X)$.

**Definition 4.4.** We say that a continuous coboundary $\psi = \varphi - \varphi \circ T$ is a revelation for $f$ if $f + \psi$ is a revealed function.

Examples of such revelations can be found in [Je4]. Notice that if $\psi$ is a weak-coboundary, then $\beta(f) = \beta(f + \psi)$ and $\mathcal{M}_T^{max}(f) = \mathcal{M}_T^{max}(f + \psi)$. So, by Proposition 4.2 we have the following proposition:

**Proposition 4.5.** Let $\psi$ be a weak-revelation for $f$. Then

1. $\beta(f) = \max(f + \psi)$;
2. $\mathcal{M}_T^{max}(f) = \{\mu \in \mathcal{M}_T : supp(\mu) \subset (f + \psi)^{-1}(\max(f + \psi))\}$.

[2] $\psi \in C(X)$ is a weak-coboundary if $\int \psi d\mu = 0 \; \forall \mu \in \mathcal{M}_T$.

There is a lot of literature concerning the existence of such revelations (which are in particular weak-revelations) [Sa, Bo1, CG, Je4]. In this note, we will discuss the existence of revelations for certain classes of dynamical systems. The most part of this chapter were from a notes written jointly with Sebastian Burgos.

## 4.2 Expandingness

In this section we will always assume that $(X, T)$ is a topological dynamical system.

**Definition 4.6.** We say that $T : X \to X$ is distance-expanding if there exist constants $\lambda > 1$ and $\eta > 0$ such that $d(Tx, Ty) \geq \lambda d(x, y)$ whenever $d(x, y) \leq 2\eta$.

**Definition 4.7.** We say that $T : X \to X$ is (topologically) transitive if for every non-empty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$.

**Remark 4.8.** Notice that if $T$ is topologically transitive, then it maps $X$ onto $X$.

**Definition 4.9.** We say that $T : X \to X$ is expanding if the following hold:

1) $T$ is distance-expanding,

2) $T$ is open,

3) $T$ is transitive.

We will see that conditions *1)* and *2)* over $T$ ensures that $T$ is locally-invertible, and these local inverse branches are contractions. Condition *3)* allows to use the *specification property* ([PU, Theorem 3.3.12]), which ensures the existence of a certain shadowing property concerning periodic orbits.

**Lemma 4.10.** ([PU, Lemma 3.1.2]) If $T$ is open, then for every $\eta > 0$, there exists $\xi > 0$ such that

$$T(B(x, \eta)) \supset B(Tx, \xi)$$

for every $x \in X$.

*Proof.* For every $x \in X$, define

$$\xi(x) := \sup\{r > 0 : T(B(x, \eta)) \supset B(Tx, r)\}.$$

Since $T$ is open, $\xi(x) > 0$. Since $T(B(x, \eta)) \supset B(Tx, \xi(x))$, it is enough to show that $\xi := \inf\{\xi(x) : x \in X\} > 0$. Assume by contradiction that $\xi = 0$, so there exists

$\{x_n\}_n \subset X$ such that $\xi(x_n) \to 0$ as $n \to \infty$. Since $X$ is compact, we can assume that $x_n \to y \in X$, and therefore

$$B(x_n, \eta) \supset B(y, \eta/2)$$

for $n$ large enough.

By the openness of $T$, there exists $\varepsilon > 0$ such that for every $n$ large enough

$$T(B(x_n, \eta)) \supset T(B(y, \eta/2)) \supset B(Ty, \varepsilon) \supset B(Tx_n, \varepsilon/2).$$

Then, $\xi(x_n) \geq \varepsilon/2$ for $n$ large enough, which is a contradiction. ■

If $T$ is distance-expanding, it follows from the definition that for every $x \in X$, the map $T|_{B(x,\eta)}$ is injective and therefore it has an inverse map on $T(B(x, \eta))$. If additionally $T$ is open, by Lemma 4.10 the domain of the inverse map around $Tx$ contains the ball $B(Tx, \xi)$.

**Definition 4.11.** Assume that $T$ is distance-expanding and open, and let $x \in X$. We denote by $T_x^{-1} : B(Tx, \xi) \to B(x, \eta)$ the inverse map of $T|_{B(x,\eta)}$ restricted to $B(Tx, \xi)$.

**Lemma 4.12.** [PU, Lemma 3.1.4] For $x \in X$ and $y, z \in B(Tx, \xi)$, we have

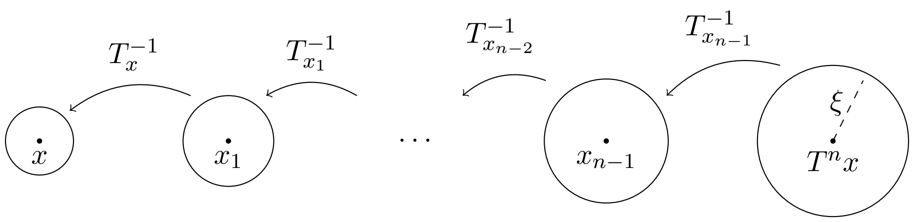
$$d(T_x^{-1}(y), T_x^{-1}(z)) \leq \lambda^{-1} d(y, z).$$

In particular $T_x^{-1}(B(Tx, \xi)) \subset B(x, \lambda^{-1}\xi) \subset B(x, \xi)$ and $T(B(x, \lambda^{-1}\xi)) \supset B(Tx, \xi)$ for $\xi > 0$ small enough (what specifies the Lemma 4.10).

**Remark 4.13.** For every $x \in X$, $n \in \mathbb{N}$ and $0 \leq j \leq n-1$ write $x_j := T^j x$. By Lemma 4.12 the composition

$$T_x^{-1} \circ T_{x_1}^{-1} \circ \cdots \circ T_{x_{n-1}}^{-1} : B(T^n x, \xi) \to X$$

is well defined.




**Theorem 4.14.** *([PU, Theorem 3.3.12]) Let $T : X \to X$ be a continuous mapping. Assume that $T$ is expanding, then the following specification property holds: For every $\beta > 0$, there exists a positive integer $N$ such that for every $n \geq 0$ and every T-orbit $(x_0, ..., x_n)$ there exists*

*a periodic point $y$ of period not large than n+N whose orbit for the times 0, ... , n $\beta$-shadows $(x_0, ..., x_n)$. Moreover, if we denote by $Per(y)$ the minimal period of $y$, then there exists $k \in \mathbb{N}$ such that $n \leq k \cdot Per(y) \leq n + N$.*

## 4.3 Mañé lemma for expanding maps

In this section, we follow Jenkinson ([Je4, Theorem 6.2]), in which he proves the existence of revelations for expansive maps and 'regular' functions. The argument is based the proof of the Manñé Lemma due to Contreras, Lopes and Thieullen [CoLT] for $C^1$-expanding maps of the circle.

**Theorem 4.15.** *([Je4, Theorem 6.2]) Let $T : X \to X$ be an expanding map. Then every $f \in Lip(X)$ has a revelation $\psi = \varphi - \varphi \circ T$, with $\varphi \in Lip(X)$.*

*Proof.* First, we prove the following characterization of being a revelation for $f$:

**Lemma 4.16.** A continuous coboundary $\psi$ is a revelation for $f$ if and only if $f + \psi \leq \beta(f)$.

*Proof.* First suppose that $\psi$ is a revelation for $f$. Then, since $f + \psi$ is a revealed function,

$$f + \psi \leq \max(f + \psi) = \beta(f + \psi) = \beta(f).$$

Conversely, assume that $f + \psi \leq \beta(f)$ and let $\mu \in \mathcal{M}_T^{max}(f) = \mathcal{M}_T^{max}(f + \psi)$ (since $\psi$ is a coboundary). Then $\int(f + \psi)d\mu = \beta(f + \psi) = \beta(f)$ and thus

$$\int \underbrace{(\beta(f) - (f + \psi))}_{\geq 0} d\mu = 0.$$

This implies that

$$supp(\mu) \subset \{x \in X : (f + \psi)(x) = \beta(f)\}. \tag{4.1}$$

This last set is non-empty, since $\mu$ is a probability measure, and hence $\max(f + \psi) = \beta(f)$. Notice that (4.1) can be written as $supp(\mu) \subset (f + \psi)^{-1}(\max(f + \psi))$.

We claim that $supp(\mu)$ is a compact $T$-invariant set. The compactness is clear, since it is a closed set on a compact metric space. In order to prove that $supp(\mu)$ is $T$-invariant, take

$x \in supp(\mu)$ and $\varepsilon > 0$. It is enough to prove that $\mu(B(Tx,\varepsilon)) > 0$. By the continuity of $T$, there exists $\delta > 0$ such that $T(B(x,\delta)) \subset B(Tx,\varepsilon)$. Then, since $\mu$ is $T$-invariant

$$0 < \mu(B(x,\delta)) \leq \mu(T^{-1}B(Tx,\varepsilon)) = \mu(B(Tx,\varepsilon)).$$

Hence, $f+\psi$ is revealed and the proof of Lemma 4.16 is finished. ■

Now denote by $S_n f := \sum_{j=0}^{n-1} f \circ T^j$ the $n$th Birkhoff sum of $f$. For $x \in X$ define

$$\varphi(x) := \sup_{n\geq 1} \sup_{y \in T^{-n}x} (S_n f(y) - n\beta(f)). \tag{4.2}$$

**Remark 4.17.** Since $T$ is transitive, it is surjective so the set $T^{-n}x$ is non-empty for every $x \in X$ and $n \geq 1$. To prove that $\varphi$ is well defined, it only remains to show that $\varphi(x) < \infty$ for every $x \in X$. Also notice that we can assume without loss of generality that $\beta(f) = 0$ (otherwise set $\tilde{f} := f - \beta(f)$).

**Lemma 4.18.** For every $x \in X$, $\varphi(x) < \infty$.

*Proof.* Let $x \in X$, and consider $N \in \mathbb{N}$ given by the specification property (see Theorem 4.14, using $\beta = \xi$ of the Lemma 4.10). For $n \geq N$, take $y \in T^{-n}x$. Then, there exists a periodic point $p \in X$ with $n - N \leq per(p) \leq n$ (set $k := per(p)$, and notice that it is not necessarily the minimal period of $p$) such that $d(T^j y, T^j p) \leq \xi$ for $0 \leq j \leq n-N$. Therefore, since $T$ is expanding we have

$$\begin{aligned}
|S_n f(y) - S_k f(p)| &\leq \left| \sum_{j=0}^{n-N} (f(T^j y) - f(T^j p)) \right| + \left| \sum_{j=n-N+1}^{k-1} (f(T^j y) - f(T^j p)) \right| + \left| \sum_{j=k}^{n-1} f(T^j y) \right| \\
&\leq Lip(f) \sum_{j=0}^{n-N} d(T^j y, T^j p) + 2\|f\|_\infty (\underbrace{k-n}_{\leq 0}+N) + (\underbrace{n-k}_{\leq N}) \|f\|_\infty \\
&\leq Lip(f) d(T^{n-N}y, T^{n-N}p) \sum_{j=0}^{n-N} \lambda^{-(n-N-j)} + 3N\|f\|_\infty \\
&\leq \frac{Lip(f)\operatorname{diam} X}{1-\lambda^{-1}} + 3N\|f\|_\infty =: C
\end{aligned}$$

Thus

$$S_n f(y) \leq C + S_k f(p) \leq C,$$

since

$$\frac{1}{k} S_k f(p) = \int f d \underbrace{\left( \frac{1}{k} \sum_{j=0}^{k-1} \delta_{T^j p} \right)}_{\in \mathcal{M}_T} \leq \beta(f) = 0.$$

Now taking supremum over $y \in T^{-n}x$ and then over $n \geq N$ we get $\varphi(x) < \infty$. ■

To prove that $\varphi - \varphi \circ T$ is a revelation for $f$, let $x \in X$. Since $T^{-(n-1)}(x) \subset T^{-n}(Tx)$,

$$\varphi(Tx) = \sup_{n\geq 1} \sup_{y \in T^{-n}(Tx)} S_n f(y) \geq \sup_{n \geq 1} \sup_{y \in T^{-(n-1)}(x)} S_n f(y). \tag{4.3}$$

Now if $y \in T^{-(n-1)}(x)$, then $S_n f(y) = f(x) + S_{n-1} f(y)$ for every $n \geq 1$ (with $S_0 f \equiv 0$). So by (4.3)

$$\varphi(Tx) \geq f(x) + \sup_{n\geq 1} \sup_{y \in T^{-(n-1)}(x)} S_{n-1} f(y).$$

However,

$$\sup_{n\geq 1} \sup_{y \in T^{-(n-1)}(x)} S_{n-1} f(y) = \sup_{N \geq 0} \sup_{T^{-N}(x)} S_N f(y) \geq \sup_{N\geq 1} \sup_{T^{-N}(x)} S_N f(y) = \varphi(x),$$

which implies that $f(x) + \varphi(x) - \varphi(Tx) \leq 0 = \beta(f)$. By Lemma 4.16 we conclude that $\varphi - \varphi \circ T$ is a revelation for $f$.

Now, to prove that $\varphi$ is Lipschitz we will need the following Lemma:

**Lemma 4.19.** Let $g : X \to \mathbb{R}$ be a function. If there exist constants $C, \varepsilon > 0$ such that for every $x \in X$, $g|_{B(x,\varepsilon)}$ is Lipschitz with $Lip(g|_{B(x,\varepsilon)}) \leq C$, then $g \in Lip(X)$.

*Proof.* First, there exists $M > 0$ such that $|g| \leq M$. So given $x, y \in X$ two different points, we have two cases: if $d(x, y) < \varepsilon$, then

$$\frac{|g(x) - g(y)|}{d(x,y)} \leq C.$$

Otherwise, if $d(x, y) \geq \varepsilon$, then

$$\frac{|g(x) - g(y)|}{d(x,y)} \leq \frac{2M}{\varepsilon}.$$

Therefore, $g$ is Lipschitz. ■

Now we will prove that $\varphi$ defined in (4.2) is Lipschitz.

Let $B$ be a ball of radius $\xi$. Let $x, x' \in B$ and $\varepsilon$. By definition of $\varphi$, there exist $N \geq 1$ and $y \in T^{-N}(x)$ such that

$$\varphi(x) \leq \varepsilon + S_N f(y). \tag{4.4}$$

Since $T$ is expanding, by Remark 4.13 we can write $y = T_{i_1}^{-1} \circ \cdots \circ T_{i_N}^{-1}(x)$, where $T_{i_j}^{-1}$ denote inverse branches of $T$ (recall that we are in a ball of radius $\xi$). Define

$$y' := T_{i_1}^{-1} \circ \cdots \circ T_{i_N}^{-1}(x').$$

In particular $y' \in T^{-N}(x')$, so $S_N f(y') \leq \sup_{z \in T^{-N}(x')} S_N f(z)$ and therefore

$$S_N f(y') \leq \sup_{n \geq 1} \sup_{z \in T^{-n}(x')} S_N f(z) = \varphi(x'). \tag{4.5}$$

Thus, combining (4.4) and (4.5) we have $\varphi(x) - \varphi(x') \leq S_N f(y) - S_N f(y') + \varepsilon$. However,

$$\begin{aligned} S_N f(y) - S_N f(y') &= \sum_{j=0}^{N-1} (f(T^j y) - f(T^j y')) \leq Lip(f) \sum_{j=0}^{N-1} d(T^j y, T^j y') \\ &\leq Lip(f) \sum_{j=0}^{N-1} \lambda^{-(N-j)} d(x, x') \leq \frac{Lip(f)}{1 - \lambda^{-1}} d(x, x'). \end{aligned}$$

Since $\varepsilon$ was arbitrary we proved that

$$\varphi(x) - \varphi(x') \leq \frac{Lip(f)}{1 - \lambda^{-1}} d(x, x').$$

So $\varphi|_B$ is Lipschitz with $Lip(\varphi|_B) \leq \dfrac{Lip(f)}{1 - \lambda^{-1}}$ for every ball $B$ of radius $\xi$. By Lemma 4.19 we conclude that $\varphi \in Lip(X)$ and the proof of Theorem 4.15 is finished. ■

## 4.4 Bousch's bilateral Mañé lemma

Let $(X, T)$ be a topological dynamical system and let $f : X \to \mathbb{R}$ be a continuous function which has a revelation cohomologous to $f$, that is, there exists a continuous function $\varphi : X \to \mathbb{R}$ such that:

$$f + \varphi - \varphi \circ T \leq \beta(f). \tag{4.6}$$

Suppose that $-f$ also has a revelation which is cohomologous to $-f$, that is, there exists a continuous function $\psi : X \to \mathbb{R}$ such that:

$$f + \psi - \psi \circ T \geq \alpha(f), \tag{4.7}$$

where $\alpha(f)$ is the ergodic infimum of $f$. In this section, we prove a Bousch's theorem which states that we can take a common revelation satisfying both inequalities (4.6) and (4.7).

**Theorem 4.20.** *([Bo2, Theorem 1]) Let $T : X \to X$ be a continuous map in a compact metric space and $f : X \to \mathbb{R}$ a continuous function. Suppose that there exists two functions $\varphi, \psi : X \to \mathbb{R}$ satisfying (4.6) and (4.7). Then, if $\alpha(f) < \beta(f)$, there exists a continuous function $\rho : X \to \mathbb{R}$ such that:*

$$\alpha(f) \leq f + \rho - \rho \circ T \leq \beta(f). \tag{4.8}$$

*If $\alpha(f) = \beta(f)$ and $T$ is transitive, then there exists $\rho : X \to \mathbb{R}$ satisfying (4.8). In other words, if $f$ has cohomologous revelations on both sides, then there exists a bilateral revelation.*

*Proof.* We divide the proof into two cases. Suppose that $\alpha(f) = \beta(f)$. Define

$$g = f + \psi - \psi \circ T - \alpha(f) \text{ and } \phi = \varphi - \psi.$$

Thus $g \geq 0$ and $g + \phi - \phi \circ T \leq 0$. Combining these two inequalities we have that

$$\phi \circ T \geq \phi + g \geq \phi.$$

Therefore $\phi \circ T \geq \phi$. The transitivity of $T$ implies that $\phi$ is constant, which is equivalent to the fact that $f$ is co-homologous to $\beta(f)$, and we are done in this case taking $\rho = \varphi$. Suppose that $\alpha(f) < \beta(f)$. Withouth any loss of generality suppose that there exists $C > 0$ such that $0 \leq \psi - \phi \leq C$. Define the revealed functions

$$\phi_\alpha = f + \psi - \psi \circ T \text{ and } \phi_\beta = f + \varphi - \varphi \circ T.$$

Also, define a continuous function $h : X \to \mathbb{R}$ by

$$h = \nu\phi_\beta + (1 - \nu)\phi_\alpha,$$

for a certain continuous function $\nu : X \to \mathbb{R}$ that we are going to take later. We claim that we can choose $\nu$ such that a very large Birkhoff average of $h$ is revealed by both sides, that is, there exists $n \in \mathbb{N}$ satisfying that

$$\alpha(f) \leq \frac{h^{(n)}}{n} \leq \beta(f).$$

For this purpose, fix $n \in \mathbb{N}$ and we calculate the Birkhoff sum of $h$:

$$h^{(n)} = f^{(n)} + (1 - \nu)\psi - (1 - \nu)\psi \circ T^{n+1} + \nu\varphi - \nu\varphi \circ T^{n+1}.$$

From the above equality, we have that

$$h^{(n)} \geq \phi_{\alpha}^{(n)} + \nu(\varphi - \psi), \tag{4.9}$$

and

$$h^{(n)} \leq \phi_{\beta}^{(n)} + (1-\nu)(\psi - \varphi). \tag{4.10}$$

Note that if $\nu = 0$ in 4.9 implies that $h^{(n)} \geq \alpha(f) \cdot n$, and also if $\nu = 1$ in 4.10 implies that $h^{(n)} \leq \beta(f) \cdot n$. Therefore we need to take a very appropriate $\nu$ in order to have both inequalities valid in the whole space $X$. For this purpose, it is convenient to define the compact sets:

$$K_n = \left\{ x \in X : (\phi_{\alpha}^{(n)} + \nu(\phi - \psi))(x) < \alpha(f) \cdot n \right\},$$

$$L_n = \left\{ x \in X : (\varphi_{\beta}^{(n)} + (1-\nu)(\psi - \varphi))(x) > \beta(f) \cdot n \right\}.$$

It is not difficult to see that for large $n$ we have that $K_n \cap L_n = \emptyset$. Take such $n$ and a continuous $\nu : X \to [0,1]$ such that $\nu|_{K_n} = 0$ and $\nu|_{L_n} = 1$. Therefore

$$\forall x \in X \implies \alpha(f) \cdot n \leq h^{(n)}(x) \leq \beta(f) \cdot n,$$

which is what we claimed.
Since $h$ is cohomologous to both $f$ and $\dfrac{h^{(n)}}{n}$, there exists a continuous $\rho : X \to \mathbb{R}$ such that:

$$\alpha(f) \leq f + \rho - \rho \circ T \leq \beta(f). \qquad \blacksquare$$

# Chapter 5

# Generic Rotation Sets

## 5.1 Introduction

Let $(X, T)$ be a topological dynamical system. Given a continuous vector valued potential $F : X \to \mathbb{R}^d$, we define its *rotation set* as:

$$R(F) = \left\{ \int F \, \mathrm{d}\mu : \mu \in \mathcal{M}_T \right\}.$$

This is a convex body in $\mathbb{R}^d$, that is, a non-empty compact and convex subset of $\mathbb{R}^d$.

This definition originates from the rotation theory on the torus [MK]: if $f : \mathbb{T}^d \to \mathbb{T}^d$ is continuous, homotopic to the identity with lift $\widetilde{f} : \mathbb{R}^d \to \mathbb{R}^d$, we consider the displacement function $F(x) := \widetilde{f}(x) - x$. The corresponding rotation set $R(F)$ yields important information about the dynamics of $f$. Note that in the one-dimensional case, $R(F) = \{\rho(\widetilde{f})\}$, where $\rho(\cdot)$ is the Poincaré rotation number. For $d \geq 2$, it is known that generically the rotation set is given by a rational polygon [P], and there are rotation sets with smooth boundary points [BCH].

Returning to the general context, Ziemian [Zi] studied the situation where the dynamics is a subshift of finite type (SFT) and the potential $F$ is locally constant, and proved that in this case the rotation set is a polytope. On the other hand, Kucherenko and Wolf [KW] proved that if $T$ is a SFT then every convex body of $\mathbb{R}^d$ appears as a rotation set of a continuous potential.

Ergodic optimization is another motivation for the study of the rotation set. Recall that in the one-dimensional case, the rotation set is $R(f) = [\alpha(f), \beta(f)]$.

Consider the more general problem of computing the maximum ergodic average $\beta(f)$ for all functions $f$ in a given finite-dimensional subspace of $C(X)$, say with generators

$f_1, ..., f_d$. If $f = \sum_{j=1}^{d} \alpha_j f_j$ then we have:

$$\beta(f) = \sup_{\vec{x} \in R(F)} (\alpha_1, ..., \alpha_d) \cdot \vec{x}$$

where $F = (f_1, f_2, ..., f_d)$. Therefore, the problem reduces to the study of the rotation set of $F$, which is called Vectorial Ergodic Optimization [B, section 2].

Let us describe one of the first examples of rotation sets, introduced by Jenkinson [Je1]. Let $X = \mathbb{R}/\mathbb{Z}$ be the circle, $T$ be the doubling map, and $F(x) = (\cos(2\pi x), \sin(2\pi x))$ be the potential. The corresponding rotation set $R(F)$ is called *the fish*. Validating experimental results of Jenkinson, Bousch [Bo1] proved that the fish is strictly convex and every point on its boundary is the integral of $F$ with respect to a unique $T$-invariant probability measure. Furthermore, he proved that the fish has a dense subset of corners (points of non-differentiability), and each corner is the integral of $F$ with respect to a unique $T$-invariant porbability measure, which is *periodic*, that is, supported on a single periodic orbit.

It is natural to ask whether these characteristics of the fish are typical among rotation sets: see [B, section 2] for further discussion. In this work, we give a partial answer to this question. Under a mild hypothesis on the dynamics $T$ (which is satisfied for the doubling map and SFT), we prove that for generic continuous potentials, the rotation set is strictly convex and (unlike the fish) has a $C^1$ boundary. This genericity result is obtained as a corollary of our main theorem, which reads as follows:

**Theorem 5.1.** *Let $T : X \to X$ be a non-uniquely ergodic topological dynamical system with dense set of periodic measures. Then the map*

$$R : (C(X, \mathbb{R}^d), \|\cdot\|_\infty) \to (CB(\mathbb{R}^d), d_H)$$

*that associates to each potential $F$ its rotation set $R(F)$ is continuous, open, and surjective.*

Here, $C(X, \mathbb{R}^d)$ is endowed with the uniform norm, and $CB(\mathbb{R}^d)$ is the set of convex bodies of $\mathbb{R}^d$ endowed with the Hausdorff distance (see section 5.2 for more details). Continuity of the map $R$ is trivial. Surjectivity of $R$ was already known when $T$ is a SFT: see [KW, Theorem 2].

The hypothesis of denseness of periodic measures holds for any dynamical system with the specification property (e.g., uniformly expanding transformations, SFT, and Anosov diffeomorphisms). It also holds for many classes of non-hyperbolic dynamics, for example, $\beta$ shifts, $S$-gap shifts, and isolated non-trivial transitive sets of $C^1$-generic diffeomorphisms: see [GK].

As a consequence of our main result, we have:

**Corolary 5.2.** Let $T : X \to X$ be a non-uniquely ergodic topological dynamical system with dense set of periodic measures. Then there exists a residual subset $\mathcal{R}$ of $C(X, \mathbb{R}^d)$ such that $R(F)$ is strictly convex and has $C^1$ boundary for all $F \in \mathcal{R}$.

*Proof of Corollary 5.2.* The set of convex bodies which are strictly convex with $C^1$ boundary is residual [S, p. 133]. Therefore the pre-image under $R$ of this set is also residual, since $R$ is continuous and open by Theorem 5.1. ■

The $C^1$ regularity in the corollary cannot be improved in this case: for generic convex bodies the boundary is not $C^{1+\alpha}$, for any $\alpha > 0$ : see [KliN].

It is natural to ask whether Theorem 5.1 holds for spaces of more regular functions, for example, Lipschitz functions. The answer is negative: see section 5.5.2.

## 5.2 Preliminaries

Let $A \subset \mathbb{R}^d$ and $\varepsilon > 0$. We define $\varepsilon$-neighbourhood of $A$ as

$$B_\varepsilon(A) = \{x \in \mathbb{R}^d : \inf_{y \in A} \|x - y\| < \varepsilon\}.$$

We say that a non-empty subset $K \subset \mathbb{R}^d$ is a *convex body* if it is compact and convex. We denote the set of convex bodies by $CB(\mathbb{R}^d)$, and by $CB_\circ(\mathbb{R}^d)$ the set of convex bodies with non-empty interior. Aditionally, given a convex body $K$, we denote by $\mathrm{int}(K)$ its interior and $\mathrm{relint}(K)$ its relative interior. We endow $CB(\mathbb{R}^d)$ with a structure of metric space, given by the *Hausdorff distance* defined by:

$$d_H(K, L) = \max\left\{\sup_{x \in K} \inf_{y \in L} \|x - y\|, \sup_{y \in L} \inf_{x \in K} \|x - y\|\right\}.$$

This definition only requires $K, L$ to be compact. Also, the $\sup$ and $\inf$ can be replaced by $\max$ and $\min$ due to compactness. This metric turns $CB(\mathbb{R}^d)$ into a complete, locally compact metric space [S, p. 62]. The Hausdorff distance between two convex bodies can also be obtained just considering their boundaries, namely, if $K, L$ are two convex bodies, then $d_H(K, L) = d_H(\partial K, \partial L)$ [S, p. 61]. A useful lemma that will be used later is the following:

**Lemma 5.3.** *Let $K \in CB(\mathbb{R}^d)$ and $0 < \delta < 1$. Then there exists $K_\delta \in CB(\mathbb{R}^d)$ such that $K_\delta \subset \mathrm{relint}(K)$ and $d_H(K_\delta, K) < \delta$.*

*Proof.* Define the *support function* of an arbitrary $L \in CB(\mathbb{R}^d)$ by

$$h_L(u) = \sup_{x \in L} x \cdot u,$$

and denote $\overline{h}_L = h_L|_{\mathbb{S}^{n-1}}$. Applying a translation if necessary, suppose that $0 \in \operatorname{relint}(K)$. Define $K_\delta = \left(1 - \frac{\delta}{kd}\right) K$, where $d = \sup_{x \in \partial K} \|x\|$ and $k \in \mathbb{N}$ is such that $\frac{\delta}{kd} < 1$. It is clear that $K_\delta \subset \operatorname{relint}(K)$, and using [S, Lemma 1.8.14] :

$$\begin{aligned} d_H(K_\delta, K) &= \left\|\overline{h}_{K_\delta} - \overline{h}_K\right\|_\infty \\ &\leq \frac{\delta}{kd} \left\|\overline{h}_K\right\|_\infty \\ &= \frac{\delta}{kd} \sup_{x \in \partial K} \|x\| < \delta. \end{aligned}$$

■

Let $X$ be a compact metric space and let $T : X \to X$ be a continuous map. Given $x \in X$, we denote by $\mathcal{O}(x) = \{T^j(x) : j \geq 0\}$ its positive orbit. For a periodic point $x \in X$, we denote by $\mu_{\mathcal{O}(x)}$ the unique $T$-invariant probability measure supported in $\mathcal{O}(x)$. These measures are called *periodic*, and $\mathcal{M}_T^{\text{per}}$ denotes the set of periodic measures.

Letting $F : X \to \mathbb{R}^d$ be a continuous potential, we use the following notation for Birkhoff sums:

$$F^{(n)} := F + F \circ T + ... + F \circ T^{n-1}.$$

Recall that the *rotation set* of $F$ is defined as:

$$R(F) = \left\{ \int F \, \mathrm{d}\mu : \mu \in \mathcal{M}_T \right\}.$$

This is a compact convex subset of $\mathbb{R}^d$. Also, define the *periodic rotation set* of $F$ as:

$$R_{\text{per}}(F) = \left\{ \int F \, \mathrm{d}\mu : \mu \in \mathcal{M}_T^{\text{per}} \right\}.$$

Clearly if $\mathcal{M}_T^{\text{per}}$ is dense in $\mathcal{M}_T$, then $R_{\text{per}}(F)$ is dense in $R(F)$. Let us prove the continuity of the map $R$:

**Proposition 5.4.** The map $R : (C(X, \mathbb{R}^d), \|\|_\infty) \to CB(\mathbb{R}^d, d_H)$ is continuous.

*Proof.* Let $\mu \in \mathcal{M}_T$ and $F, G \in C(X, \mathbb{R}^d)$, and note that:

$$\left\| \int F \, \mathrm{d}\mu - \int G \, \mathrm{d}\mu \right\| \leq \|F - G\|_\infty$$

and this immediately implies that $d_H(R(F), R(G)) \leq \|F - G\|_\infty$. ■

## 5.3 Approximate Mañé Lemma

The *Mañé lemma* is a useful tool in ergodic optimization [Sa, Bo1, CG, Je4, Bo2]. It is stated as follows in the particular situation of expanding dynamics: let $T : X \to X$ be a expanding map and $\alpha \in (0, 1]$. Then, for any $f$ in the space $C^{\alpha}(X)$ of $\alpha$-Hölder functions, there exists $h \in C^{\alpha}(X)$ such that $\alpha(f) \leq f + h \circ T - h \leq \beta(f)$, where $\alpha(f)$ and $\beta(f)$ are the minimum and maximum ergodic average, respectively. This says that up to adding a coboundary $h - h \circ T$ to $f$ (which does not alter the integrals with respect invariant measures), we can assume that the image of $f$ is contained in the rotation set $R(f) = [\alpha(f), \beta(f)]$.

We can ask if there is an analogous of the Mañé Lemma in the setting of vectorial potentials. Following the same spirit of the Mañé Lemma, we say that a vectorial potential $F \in C(X, \mathbb{R}^d)$ satisfies the *Mañé Lemma* if there exists $H \in C(X, \mathbb{R}^d)$ such that $\mathrm{Im}(F + H - H \circ T) \subset R(F)$. Even if we impose some regularity on $F$, the classical example of the fish is a Hölder function that does not satisfy the Mañé Lemma, as noted by Bochi and Delecroix: see [B, Proposition 2.1].

Nevertheless, we have the following *approximate Mañé Lemma*:

**Lemma 5.5.** *Let $F : X \to \mathbb{R}^d$ be a continuous function and $\varepsilon > 0$. Then there exists a continuous function $G : X \to \mathbb{R}^d$ cohomologous to $F$ such that:*

$$\mathrm{Im}\,(G) \subset B_{\varepsilon}(R(F)).$$

*Moreover, there exists $N_0 \in \mathbb{N}$ such that $G$ can be taken to be $\frac{1}{n}F^{(n)}$ for arbitrary $n \geq N_0$.*

Lemma 5.5 is well known (c.f. "enveloping property" [B, p. 6]), but for completeness we give a proof. Note that if $F : X \to \mathbb{R}^d$ is continuous, then $F$ is cohomologous to $\frac{1}{n}F^{(n)}$ for all $n \in \mathbb{N}$, since $F = \frac{1}{n}F^{(n)} + H - H \circ T$, where $H = \frac{1}{n}\sum_{j=1}^{n} F^{(j)}$.

*Proof of Lemma 5.5.* Suppose in order to get a contradiction that there exists a number $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that:

$$\frac{1}{n}F^{(n)}(x_n) \notin B_{\varepsilon}(R(F)).$$

Consider the following sequence of probability measures on $X$:

$$\mu_n = \frac{\delta_{x_n} + \delta_{T(x_n)} + .. + \delta_{T^{n-1}(x_n)}}{n}.$$

By compactness of the space of probability measures there exists a subsequence $\mu_{n_k}$ converging to a probability measure $\mu$. It is not hard to see that $\mu$ is a $T$-invariant probability measure. Thus, by the weak-$*$ convergence, we obtain:

$$\int F \, \mathrm{d}\mu_{n_k} \to \int F \, \mathrm{d}\mu$$

and since $B_\varepsilon(R(F))^c$ is closed, we have $\int F \, \mathrm{d}\mu \notin B_\varepsilon(R(F))$, a contradiction. Since $F$ is cohomologous to its finite time averages, we can take $G = \frac{1}{n} F^{(n)}$ for sufficiently large $n$. ■

## 5.4 Proof of the main result

In this section we present the main technical ingredients in the proof of Theorem 5.1 and we combine them at the end. We will always assume that $(X, T)$ is a non-uniquely ergodic topological dynamical system with dense set of periodic measures. The first technical lemma enlarges the rotation sets, without losing the control of the distance to the original potential. .

**Lemma 5.6.** *Let $F : X \to \mathbb{R}^d$ be a continuous function with $\sharp R(F) \geq 2$ and $K \in CB_\circ(\mathbb{R}^d)$ such that $R(F) \subset \operatorname{int}(K)$. Let $z_1, ..., z_m$ be distinct points in $R_{\text{per}}(F) \backslash \partial R(F)$ and $y_1, ..., y_m \in \operatorname{int}(K) \setminus R(F)$ be such that $R(F) \subset \operatorname{conv}\{y_1, ..., y_n\}$ (see Figure 5.1). Then there exists a continuous potential $G : X \to \mathbb{R}^d$ with:*

1. $\|G - F\|_\infty \leq \frac{7}{6} \max_i \|z_i - y_i\|$, *and*
2. $\operatorname{conv}\{y_1, ..., y_m\} \subset R(G) \subset \operatorname{int}(K)$.

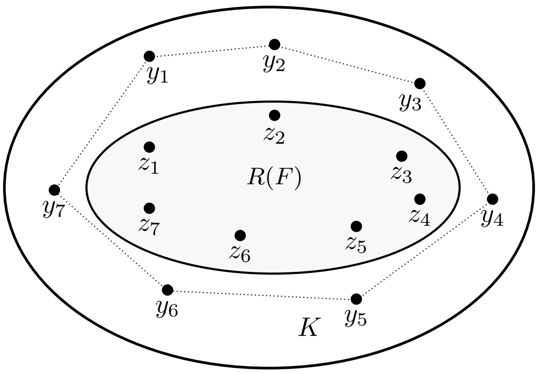



FIGURE 5.1: Setting for Lemma 5.6 with $m = 7$.

*Proof.* Fix $\varepsilon > 0$ such that $B_\varepsilon(R(F)) \subset \operatorname{int}(K)$ and $y_j \notin B_\varepsilon(R(F))$ for all $j = 1, ..., m$. Thus, we can apply Lemma 5.5 with the set of points $\{y_1, ..., y_m\}$ to obtain $n \in \mathbb{N}$ such that:

- $\mathrm{Im}\left(\frac{1}{n}F^{(n)}\right) \subset B_\varepsilon(R(F)) \subset \mathrm{int}(K)$
- For each $z \in \{z_1, ..., z_m\}$ there exists a periodic point $x \in X$ such that the Birkhoff average $\frac{1}{n}F^{(n)}$ equals $z$ on the orbit of $x$. We denote by $x_j$ the corresponding point to $z_j$. For this we choose $n$ sufficiently large with $n$ multiple of $\mathrm{lcm}(\sharp\mathcal{O}(x_1), ..., \sharp\mathcal{O}(x_m))$.

The main idea is to perturb the potential $F$ nearby the periodic orbits. For this purpose, let us choose the index set $I = \{(i,j) : 1 \le i \le m, 1 \le j \le \sharp\mathcal{O}(x_i)\}$ and a collection of open balls $\{B_{(i,j)}\}_{(i,j)\in I} \subset X$ centered at the periodic points defined by:

$$B_{(i,j)} = B_r(T^j(x_i)) \quad \forall (i,j) \in I$$

and $r > 0$ sufficiently small so that the collection of balls $\{B_{(i,j)}\}_{(i,j)\in I}$ are pairwise disjoint, $\frac{1}{n}F^{(n)}(B_{i,j}) \subset \mathrm{int}(R(F))$ and:

$$\text{diam}\left(\mathrm{conv}\left\{\frac{1}{n}F^{(n)}(B_{i,j}) \cup \{y_i\}\right\}\right) \le \frac{7}{6}\|y_i - z_i\| \tag{5.1}$$

Let $B_*$ be the complement of $\mathcal{O}(x_1) \cup ... \cup \mathcal{O}(x_m)$. Take a continuous partition of unity

$$\rho_* + \sum_{(i,j)\in I} \rho_{i,j} = 1$$

subordinated to the open cover $B_* \cup \bigcup_{(i,j)\in I} B_{(i,j)} = X$. Next, we define a function $\widetilde{G} : X \to \mathbb{R}^d$ as:

$$\widetilde{G}(x) = \sum_{(i,j)\in I} \rho_{i.j}(x) y_i + \rho_*(x)\frac{1}{n}F^{(n)}(x).$$

We claim that $\widetilde{G}$ satisfies similar properties as in the statement of the lemma. First, note that $\widetilde{G}$ is constant equal to $y_i$ on $\mathcal{O}(x_i)$. which implies $y_i \in R(\widetilde{G})$ for every $i = 1, ..., m$. Therefore, $\mathrm{conv}\{y_1, ..., y_m\} \subset R(\widetilde{G})$. Now,

$$\forall x \in X, \quad \widetilde{G}(x) \in \mathrm{conv}\left\{\{y_1, ..., y_m\} \cup \mathrm{Im}\left(\frac{1}{n}F^{(n)}\right)\right\},$$

since $\widetilde{G}$ is a convex combination of $y_1, ..., y_m$ and $\frac{1}{n}F^{(n)}$. The later implies:

$$R(\widetilde{G}) \subset \mathrm{conv}\left\{\{y_1, ..., y_m\} \cup \mathrm{Im}\left(\frac{1}{n}F^{(n)}\right)\right\} \subset \mathrm{int}K.$$

Consequently, $R(F) \subset \mathrm{conv}\{y_1, ..., y_m\} \subset R(\widetilde{G}) \subset \mathrm{int}(K)$. The next step is to estimate the distance between $\widetilde{G}$ and $\frac{1}{n}F^{(n)}$. Let $x \in X$:

- If $x \in B_{i,j}$ then $\widetilde{G}(x) = \rho_{i,j}(x)y_i + (1-\rho_{i,j}(x))\frac{1}{n}F^{(n)}(x)$, and therefore, using (5.1)

$$\left\| \widetilde{G}(x) - \frac{1}{n}F^{(n)}(x) \right\| = |\rho_{i,j}(x)| \left\| y_i - \frac{1}{n}F^{(n)}(x) \right\| \leq \frac{7}{6} \|y_i - z_i\| .$$

- If $x \notin \bigcup_{(i,j)\in I} B_{i,j}$, then $\widetilde{G}(x) = \frac{1}{n}F^{(n)}(x)$.

We conclude that $\left\| \widetilde{G} - \frac{1}{n}F^{(n)} \right\|_\infty \leq \frac{7}{6} \max_i \|z_i - y_i\|$. Now consider

$$G = \widetilde{G} + \left( F - \frac{1}{n}F^{(n)} \right) .$$

Recall that $F - \frac{1}{n}F^{(n)}$ is a coboundary. Therefore, $G$ has the same rotation set of $\widetilde{G}$, which is sandwiched between $\operatorname{conv}\{y_1, ..., y_m\}$ and $\operatorname{int}(K)$. Furthermore,

$$\|G - F\|_\infty = \left\| \widetilde{G} - \frac{1}{n}F^{(n)} \right\|_\infty \leq \frac{7}{6} \max_i \|z_i - y_i\| . \qquad \blacksquare$$

At this moment, we have a technical tool to enlarge rotation sets and control the distance between the potentials. Now we will upgrade the previous lemma also considering the distance between the convex bodies.

**Lemma 5.7.** *Let $F : X \to \mathbb{R}^d$ be a continuous function, let $K \in CB_\circ(\mathbb{R}^d)$ be such that $R(F) \subset \operatorname{int}(K)$, and let $\varepsilon = d_H(R(F), K)$. Then there exists a continuous function $G : X \to \mathbb{R}^d$ with the following properties:*

1. $R(F) \subset R(G) \subset \operatorname{int}(K)$
2. $\|G - F\|_\infty \leq \kappa\varepsilon$
3. $d_H(R(G), K) \leq \kappa\varepsilon$

*where $\kappa = \frac{29}{30}$.*

*Proof.* The main idea is to take a polytope which is sufficiently close to $R(F)$ and then apply Lemma 5.6. But this is not sufficient to ensure that condition (3) is satisfied, so we need to enlarge the polytope in order to have the new rotation set moderately close to $K$.

First suppose that $R(F)$ is not a singleton. Fix $\delta \in (0, \frac{\varepsilon}{5})$ with $\overline{B_\delta(R(F))} \subset \operatorname{int}K$. Now, by [S, Theorem 1.8.16] we can take distinct points $y_1, ..., y_\ell \in B_\delta(R(F)) \setminus R(F)$ such that

$R(F) \subset \operatorname{conv}\{y_1, ..., y_\ell\}$. Due to the compactness of $\partial K$, we may choose distinct points $w_{\ell+1}, ..., w_m \in \partial K$ such that:

$$\partial K \subset \bigcup_{j=\ell+1}^{m} B_{\frac{\varepsilon}{4}}(w_j).$$

Hence, since $d_H(R(F), K) = \varepsilon$, there exist distinct points $y_{\ell+1}, ..., y_m \in \operatorname{int}(K) \setminus R(F)$ with $\|y_j - w_j\| \leq \frac{\varepsilon}{3}$ and $d(R(F), y_j) \leq \frac{2\varepsilon}{3}$ for each $j = \ell + 1, ..., m$. Since $R_{\text{per}}(F)$ is dense in $R(F)$ which by assumption is not a singleton, we can also find distinct points $z_1, ..., z_m \in R_{\text{per}}(F) \setminus \partial R(F)$ such that:

$$\|z_j - y_j\| \leq \frac{4\varepsilon}{5}$$

for all $j = 1, ..., m$. By Lemma 5.6, we can perturb $F$, and obtain a continuous $G : X \to \mathbb{R}^d$ such that:

$$\|G - F\|_\infty \leq \frac{7}{6} \cdot \frac{4\varepsilon}{5} = \frac{28\varepsilon}{30}$$

and

$$R(F) \subset \operatorname{conv}(\{y_1, ..., y_m\}) \subset R(G) \subset \operatorname{int}(K).$$

So conditions (1) and (2) are satisfied. In order to check the reamining condition (3), note first that $d_H(K, R(G)) = d_H(\partial K, \partial R(G))$. Let $x \in \partial K$. Then there exists $w_j \in \partial K$ such that $w \in B_{\frac{\varepsilon}{4}}(w_j)$. So:

$$\begin{aligned} d(w, \partial R(G)) &\leq \|w - y_j\| \\ &\leq \|w - w_j\| + \|w_j - y_j\| \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{3} \\ &\leq \frac{28\varepsilon}{30}. \end{aligned}$$

Therefore $d_H(\partial K, \partial R(G)) \leq \frac{28}{30}\varepsilon$ and this implies condition (3).

For the case when $R(F)$ is a singleton, consider a continuous perturbation $F'$ of $F$ near two disjoint periodic orbits, say $\mathcal{O}(x_1)$ and $\mathcal{O}(x_2)$, such that:

- $\int F' \, \mathrm{d}\mu_{\mathcal{O}(x_1)} \neq \int F' \, \mathrm{d}\mu_{\mathcal{O}(x_2)}$
- $R(F') \subset \operatorname{int}(K)$
- $\|F - F'\|_\infty \leq 0.01\varepsilon$
- $d_H(R(F'), K) \leq 1.01\varepsilon$

and apply the same procedure as before to $F^{'}$. ∎

As a straightforward consequence of the previous lemma, we have the following proposition:

**Proposition 5.8.** Let $F : X \to \mathbb{R}^d$ be a continuous function and let $K \in CB(\mathbb{R}^d)$ such that $R(F) \subset \operatorname{relint}(K)$. Then, there exists a continuous function $G : X \to \mathbb{R}^d$ such that $\|F - G\|_\infty \leq C d_H(R(F), K)$ and $R(G) = K$, where $C = 30$.

*Proof.* We divide the proof in two cases. First suppose that $\operatorname{int}(K) \neq \emptyset$. Apply Lemma 5.7 recursively to obtain a sequence of continuous functions $F_n : X \to \mathbb{R}^d$ such that:

- $d_H(F_n, K) \leq \kappa^n d_H(R(F), K)$
- $\|F_n - F_{n+1}\|_\infty \leq \kappa^n d_H(R(F), K)$

where $F_1 = F$. Then $\{F_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and therefore converges to a continuous function $G : X \to \mathbb{R}^d$ which satisfies:

$$\|F - G\|_\infty \leq \sum_{j=1}^{\infty} \|F_{j+1} - F_j\|_\infty \leq \sum_{j=1}^{\infty} \kappa^j d_H(R(F), K) \leq 30 d_H(R(F), K).$$

By Proposition 5.4, the map $R : C(X, \mathbb{R}^d) \to CB(\mathbb{R}^d)$ is continuous, and so:

$$R(G) = R(\lim F_n) = \lim R(F_n) = K.$$

Thus, the proof of the first case is finished. Now suppose that $\operatorname{int}(K) = \emptyset$. Let $\mathcal{P}(K)$ be the least affine hyperspace passing through $K$. We can consider $F$ as a function taking values in $\mathcal{P}(K)$ and this affine hyperplane can be identified with $\mathbb{R}^\ell$, where $\ell = \dim \mathcal{P}(K)$. In this situation we can see $K$ as a subset of this $\mathbb{R}^\ell$ with $\operatorname{int}(K) \neq \emptyset$. Consequently, the proof is reduced to the first case. ∎

Now we need an adjustment in order to drop the hypothesis $R(F) \subset \operatorname{relint} K$.

**Lemma 5.9.** *Let $F : X \to \mathbb{R}^d$ be a continuous function, $K \in CB(\mathbb{R}^d)$, and $\varepsilon > 0$. Suppose that $d_H(R(F), K) \leq \varepsilon$. Then there exists a continuous function $F' : X \to \mathbb{R}^d$ with:*

1) $R(F') \subset \operatorname{relint}(K)$
2) $\|F - F'\|_\infty \leq 2\varepsilon$
3) *There exists a continuous function $F'' : X \to \mathbb{R}^d$ cohomologous to $F'$ such that $\operatorname{Im}(F'') \subset \mathcal{P}(K)$, where $\mathcal{P}(K)$ is the least affine hyperspace containing K.*

*Proof.* The strategy is similar of the proof Lemma 5.6. Apply Lemma 5.5 to $F$ and $\varepsilon > 0$ to obtain $n \in \mathbb{N}$ with $\operatorname{Im}\left(\frac{F^{(n)}}{n}\right) \in B_\varepsilon(R(F))$. Also, apply Lemma 5.3 to $K$ and $\delta = \min\{\varepsilon, \frac{1}{2}\}$ to find $L \in CB(\mathbb{R}^d)$ with $L \subset \operatorname{relint}(K)$ and $d_H(K, L) \leq \delta$. Define $F''$ as:

$$F'' = P_L\left(\frac{1}{n}F^{(n)}\right),$$

where $P_L$ is the projection onto $L$, that is, the map which sends each point of the space to its closest point in $L$. Since $P_L$ is Lipschitz, the function $F''$ is continuous. Also $R(F'') \subset \operatorname{relint}(K)$, so the next step is to estimate $d_H(R(F''), K)$. Given $y \in K$, due to the denseness of $R_{\text{per}}(\frac{1}{n}F^{(n)})$ in $R(F)$, there exists $z \in R_{\text{per}}(\frac{1}{n}F^{(n)})$ such that $\|y - z\| \leq 2\varepsilon$. Let $\mathcal{O}(x)$ be the corresponding periodic orbit. We note that:

$$\begin{aligned}
\left\| y - \int F''\, \mathrm{d}\mu_{\mathcal{O}(x)} \right\| &\leq \|y - z\| + \left\| z - \int F''\, \mathrm{d}\mu_{\mathcal{O}(x)} \right\| \\
&\leq 2\varepsilon + \left\| \int \frac{1}{n}F^{(n)} - F''\, \mathrm{d}\mu_{\mathcal{O}(x)} \right\| \\
&\leq 2\varepsilon + \int 2\varepsilon\, \mathrm{d}\mu_{\mathcal{O}(x)} \\
&\leq 4\varepsilon,
\end{aligned}$$

since $\left\|\frac{1}{n}F^{(n)} - F''\right\|_\infty \leq 2\varepsilon$. From above we get that $d_H(K, R(F'')) \leq 4\varepsilon$. Now, it suffices to consider $F' = F'' + (F - \frac{1}{n}F^{(n)})$, which is cohomologous to $F''$. Finally,

$$\|F - F'\|_\infty = \left\| F'' - \frac{1}{n}F^{(n)} \right\|_\infty \leq 2\varepsilon. \qquad \blacksquare$$

Now we are ready to prove Theorem 5.1.

*Proof of Theorem 5.1.* Let $F \in C(X, \mathbb{R}^d)$ and $\varepsilon > 0$. Let $K \in CB(\mathbb{R}^d)$ such that $d_H(K, R(F)) \leq \varepsilon$. Let $F'$ and $F''$ given by Lemma 5.9. Then apply Proposition 5.8 to $F''$ in order to obtain a continuous function $\widetilde{G} : X \to \mathbb{R}^d$ with the properties that $R(\widetilde{G}) = K$ and $\left\| F'' - \widetilde{G} \right\|_\infty \leq 4C\varepsilon$. So, we define:

$$G = \widetilde{G} + (F' - F'').$$

Hence $R(G) = K$, since $F'$ is cohomologous to $F''$. Moreover,

$$\|G - F'\|_\infty = \left\| \widetilde{G} - F'' \right\|_\infty \leq 4C\varepsilon.$$

Therefore:

$$\|F - G\|_\infty \leq \left\|F - F'\right\|_\infty + \left\|F' - G\right\|_\infty \leq 2\varepsilon + 4C\varepsilon = (2 + 4C)\varepsilon.$$

We have just proved that

$$R(\overline{B}_{(2+4C)\varepsilon}(F)) \supset \overline{B}_\varepsilon(R(F)),$$

and this inclusion implies the openness of $R$. The surjectivity follows directly from Proposition 5.8. Let $K \in CB(\mathbb{R}^d)$, $v \in \operatorname{relint}(K)$ and $F \equiv v$. Thus, applying Proposition 5.8 to $F$, we get a continuous function $G \in C(X, \mathbb{R}^d)$ such that $R(G) = K$. ■

## 5.5 Directions for further research

In this section we discuss related problems and open questions.

### 5.5.1 The uniqueness property

Let $F \in C(X, \mathbb{R}^d)$. We say that $F$ satisfies the *uniqueness property* if for each $v \in \partial R(F)$, there exists a unique $\mu \in \mathcal{M}_T$ for which $\int F \, \mathrm{d}\mu = v$. As mentioned in the introduction, in the one-dimensional case, generic functions $f \in C(X, \mathbb{R})$ satisfy the uniqueness property. So we ask:

**Question 5.10.** Is it true that generic functions $F \in C(X, \mathbb{R}^d)$ satisfy the uniqueness property?

Of course, we can replace $C(X, \mathbb{R}^d)$ for other spaces of functions. Following the proof in the one-dimensional case in [Je1, Theorem 3.2], one can show the following:

**Proposition 5.11.** The set of $F \in C(X, \mathbb{R}^d)$ which satisfy the uniqueness property is a $G_\delta$ set.

Therefore in order to give a positive answer to Question 5.10, it is sufficient to prove denseness.

### 5.5.2 The map $R(\cdot)$ is not open in general

It is natural to ask if the map $R(\cdot)$ is open if we replace $C(X, \mathbb{R}^d)$ by other spaces of functions. The answer is negative in the space of Lipschitz functions: Let $\operatorname{Lip}(X, \mathbb{R}^2)$ be the subspace of Lipschitz potentials endowed with the Lipschitz norm

$$\|f\|_{\mathrm{Lip}} = \|f\|_{\infty} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Then, we have the following:

**Proposition 5.12.** Suppose that $T$ has a fixed point $x_0$. Then there exists an open set $U \subset \mathrm{Lip}(X, \mathbb{R}^d)$ such that for all $F \in U$, $\partial R(F)$ is non-differentiable. In particular, the restriction $R|_{\mathrm{Lip}(X,\mathbb{R}^2)}$ is not open.

*Proof.* The proof follows the same spirit as [Je2, Proposition 4.12]. Let define the function $F(x) = (0, -2d(x, x_0))$ and the open set $U = B_{\frac{1}{2}}(F)$. Let $G \in \mathrm{Lip}(X, \mathbb{R}^2)$ be a Lipschitz perturbation of $F$ with $\|G\|_{\mathrm{Lip}} < \frac{1}{2}$. We claim that $(F+G)(x_0)$ is a corner of $R(F+G)$. Since the rotation map is equivariant with respect to translations, we can assume that $G(x_0) = (0,0)$. Thus,

$$(1,1) \cdot (F+G)(x) \leq -2d(x,x_0) + \sqrt{2}G(x) \leq -2d(x,x_0) + \frac{\sqrt{2}}{2} d(x,x_0) \leq 0$$

Analogously $(1,-1) \cdot (F+G)(x) \leq 0$. We conclude that $\delta_{x_0}$ is a maximizing measure for $(1, \pm 1) \cdot (F+G)$, thus:

$$\int (F+G)\, \mathrm{d}\delta_0 = (0,0)$$

is a corner for $R(F+G)$, because $R(F+G)$ contains $(0,0)$ and is contained in the cone $\{(x,y) \in \mathbb{R}^2 : y \leq -|x|\}$ with vertex $(0,0)$. Since convex bodies with $C^1$ boundary is dense, we conclude that $R|_{\mathrm{Lip}(X,\mathbb{R}^2)}$ is not open at $F$. ∎

From this proposition, we also conclude that differentiability of the rotation set boundary is not generic when we consider the space of Lipschitz functions.

### 5.5.3 Genericity result for other spaces

In this article we considered the case of continuous potentials. We propose to investigate the same question for other spaces of functions and other dynamics:

**Question 5.13.** Is it true that the rotation set is strictly convex for generic potentials in some dense subspace of $C(X, \mathbb{R}^d)$ ?

For example, replace $C(X, \mathbb{R}^d)$ by the space of $\alpha$-Hölder potentials $C^{\alpha}(X, \mathbb{R}^d)$ with the Hölder norm. Also, in view of the fish example and Proposition 5.12, it seems that if we impose regularity to the potential, then the corresponding rotation set $R(F)$ is going to have a considerable number of corners in the boundary. So, we pose the following:

**Question 5.14.** It is true that the boundary of rotation set has a (full measure) dense subset of corners for generic potentials in $C^\alpha(X, \mathbb{R}^d)$ ?

For more discussion, see [B, Section 2].

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