Angles between Oseledets spaces

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Setting

A *measurable linear 2D cocycle* consists of:

- $(\Omega, \mathcal{S}, \mu)$ – a probability space;
- $T : \Omega \to \Omega$ – an ergodic automorphism;
- $F : \Omega \to \mathrm{GL}(2, \mathbb{R})$ – a measurable function.

The *cocycle products* $F^{(n)}(\omega)$ (where $\omega \in \Omega$, $n \in \mathbb{Z}$) are defined by:

$$F^{(1)}(\omega) = F(\omega)$$
$$F^{(n+m)}(\omega) = F^{(n)}(T^m\omega)F^{(m)}(\omega)$$

F is called *log-integrable* if

$$\int_\Omega \log \underbrace{\max\left(\|F(\omega)\|, \|F(\omega)^{-1}\|\right)}_{\text{automatically } \geq 1} \, d\mu(\omega) < \infty\,.$$

Products of i.i.d. random matrices

A special case: *one-step cocycles*.

- $\Omega = A^{\mathbb{Z}}$ product space, $\mu =$ product (Bernoulli) measure $\pi^{\otimes \mathbb{Z}}$ on Ω
- $T = \sigma =$ shift on Ω
- $F : \Omega \to \mathrm{GL}(2, \mathbb{R})$ measurable such that $F(\omega)$ only depends on ω_0

Let $\nu := F_*(\mu) =$ probability measure on $\mathrm{GL}(2, \mathbb{R})$ – it contains all the relevant information.

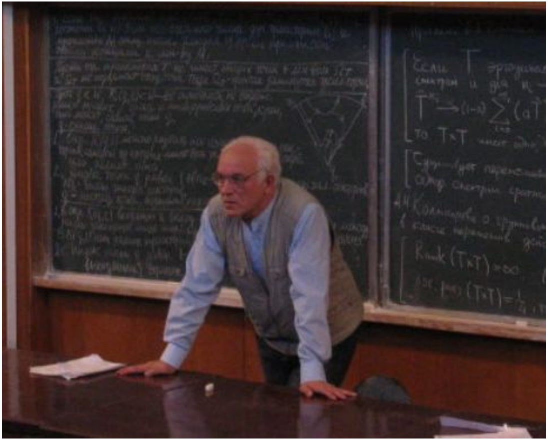
The cocycle is log-integrable iff ν has *finite first moment*:

$$\int_{\mathrm{GL}(2,\mathbb{R})} \log \max(\|g\|, \|g^{-1}\|)\, d\nu(g) < \infty\,.$$

Lyapunov exponents and Oseledets spaces

If (T, F) is a measurable log-integrable 2D cocycle, then Oseledets theorem applies, giving *Lyapunov exponents* $\lambda_1 \geq \lambda_2$ and *Oseledets spaces* (defined for μ-a.e. ω)

$$E_i(\omega) := \left\{ v \in \mathbb{R}^2 \; ; \; v = 0 \text{ or } \lim_{n \to \pm\infty} \frac{1}{n} \log \|F^{(n)}(\omega) v\| = \lambda_i \right\} .$$



Valeriy Iustinovich Oseledets

25 May 1940 – 13 March 2025

Oseledets angle $\theta := \measuredangle(E^1, E^2)$

If a cocycle has distinct Lyapunov exponents $\lambda_1 > \lambda_2$, then the Oseledets spaces are 1-dimensional and transverse:

$$\mathbb{R}^2 = E_1(\omega) \oplus E_2(\omega) \quad (\mu\text{-a.e. } \omega).$$

So the following *angle function* is defined μ-a.e.:

$$\boxed{\theta(\omega) := \measuredangle(E^1(\omega), E^2(\omega))}.$$

Angles can be small: $\operatorname{ess\,inf} \theta = 0$ is a common feature of *nonuniform hyperbolicity* (if $\lambda_1 > 0 > \lambda_2$)

How small can they be?

Two observations and a question

Proposition (Tempered property, Oseledets 1968)

The angle θ is subexponential along the orbit of μ-a.e. ω:

$$\lim_{n\to\pm\infty} \frac{1}{|n|} \log \theta(T^n\omega) = 0 .$$

Remark

$$f \in L^1(\mu) \;\Rightarrow\; \frac{f\circ T^n}{n} \to 0 \text{ a.e.}$$

Question

Is $\log \theta$ always integrable? 🤔

Answer

NO, even for one-step cocycles!

Products of i.i.d. matrices: an example and a criterion

Theorem (B.–Lessa, 2025)

*There exists a probability ν on $\mathrm{GL}(2,\mathbb{R})$ with finite **first** moment such that $\lambda_1 > \lambda_2$ and $\theta := \measuredangle(E^1, E^2)$ **is not** log-integrable.* 👎

Theorem (B.–Lessa, 2025)

*If ν is a probability ν on $\mathrm{GL}(2,\mathbb{R})$ with finite **second** moment:*

$$\int_{\mathrm{GL}(2,\mathbb{R})} \left[\log\max(\|g\|,\|g^{-1}\|)\right]^2 d\nu(g) < \infty$$

*and $\lambda_1 > \lambda_2$, then $\theta := \measuredangle(E^1, E^2)$ **is** log-integrable.* 👍

Generalization?

Question

Does the previous theorem hold for general (measurable) cocycles? 🤔

Answer

NO. In fact, no integrability condition on the cocycle is sufficient to guarantee log-integrability of the Oseledets angle function.

Flexibility of Oseledets data for measurable cocycles

Theorem (B.–Lessa, 2025)

Given:

- *An ergodic automorphism T of a non-atomic Lebesgue space $(\Omega, \mathcal{S}, \mu)$;*
- *a locally bounded function $N \colon \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R}$ s.t. $N(g) \geq \max\big(\|g\|, \|g^{-1}\|\big)$;*
- *numbers $r_1 > r_2$;*
- *a Borel probability measure η on $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1 \smallsetminus \Delta$;*

Then there exists a measurable $F \colon \Omega \to \mathrm{GL}(2,\mathbb{R})$ with $\boxed{N \circ F \in L^1(\mu)}$ s.t. the cocycle (T, F) has Lyapunov exponents $\lambda_1 = r_1$, $\lambda_2 = r_2$, and the Oseledets spaces have joint distribution η – that is, $(E_1, E_2)_* \mu = \eta$.

Bounded cocycles

Question

Can we take $F \in L^\infty$ in the previous theorem?

Answer

NO. If the cocycle is bounded, then the distribution ζ of $\log \sin \measuredangle(E_1, E_2)$ must have **bounded gaps**.

Definition

Let ζ be a Borel probability measure on $\mathbb{R}$. A *gap* of ζ is a bounded connected component of $\mathbb{R} \setminus \operatorname{supp}(\zeta)$. If the lengths of gaps of ζ are bounded above, then we say that ζ has *bounded gaps*.

Flexibility of Oseledets data for L^∞ cocycles

Bounded gaps are the only obstruction for flexibility:

Theorem (B.–Lessa, 2025)

Given:

- *An ergodic automorphism T of a non-atomic Lebesgue space $(\Omega, \mathcal{S}, \mu)$;*
- *numbers $r_1 > r_2$;*
- *a Borel probability measure η on $\mathbb{R}P^1 \times \mathbb{R}P^1 \setminus \Delta$ s.t. $(\log \sin \measuredangle)_* \eta$ has bounded gaps;*

Then there exists a bounded measurable $F \colon \Omega \to \mathrm{GL}(2, \mathbb{R})$ s.t. the cocycle (T, F) has Lyapunov exponents $\lambda_1 = r_1$, $\lambda_2 = r_2$, and the Oseledets spaces have joint distribution η.

Two general questions

Question

Take your favorite class of linear cocycles.
What can you say about the distribution of Oseledets angles?

Question

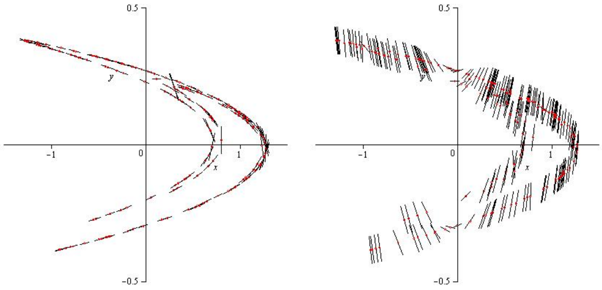
Do quantitative properties of the distribution of Oseledets angles have interesting dynamical consequences?

For Schrödinger cocycles, the regularity of the integrated density of states is related to the distribution of Oseledets angles.

Hénon map / SRB: experimental findings

$f(x, y) = (1 - ax^2 + y, bx)$ with μ = alleged SRB

Oseledets spaces E_1 and E_2 for $a = 1.4$, $b = 0.4$.

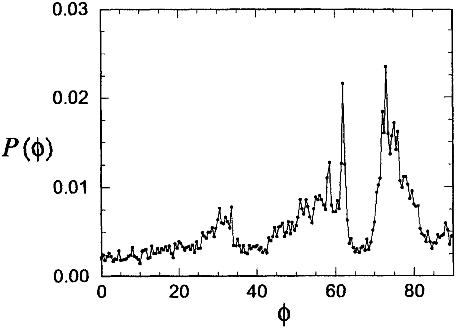


Plot by Mauro Artigiani, 2013.

Hénon map / SRB: experimental findings

$f(x, y) = (1 - ax^2 + y, bx)$ with $\mu =$ alleged SRB

Distribution of $\measuredangle(E_1, E_2)$ for $a = 1.179$, $b = 0.3$.



Plot by Anishchenko, Vadivasova, Strelkova, and Kopeikin, 1998.

MME: theoretical results

The following is a consequence of recent (Jan. 2025) work of Buzzi, Crovisier, and Sarig on exponential mixing for SPR diffeos:

Corollary

Let $f : M \to M$ be a topologically mixing C^∞ diffeomorphism of a compact surface M with $h_{\mathrm{top}}(f) > 0$. Let μ be the (necessarily unique) measure of maximal entropy. Then $\log \measuredangle(E_1, E_2) \in L^1(\mu)$ (with its distribution satisfying a power bound).

Thanks Snir Ben Ovadia for this observation.

Products of i.i.d. matrices: recap

ν = probability on $\mathrm{GL}(2,\mathbb{R})$

$$p^{\text{th}} \text{ moment of } \nu := \int_{\mathrm{GL}(2,\mathbb{R})} \big[\log \max(\|g\|, \|g^{-1}\|)\big]^p \, d\nu(g)$$

Theorem (Example of non-log-integrable angle)

There exists a probability ν on $\mathrm{GL}(2,\mathbb{R})$ with finite ***first*** *moment such that $\lambda_1 > \lambda_2$ and $\theta := \measuredangle(E^1, E^2)$* ***is not*** *log-integrable.* 👎

Theorem (Criterion of log-integrability)

If ν is a probability ν on $\mathrm{GL}(2,\mathbb{R})$ with finite ***second*** *moment and $\lambda_1 > \lambda_2$, then $\theta := \measuredangle(E^1, E^2)$* ***is*** *log-integrable.* 👍

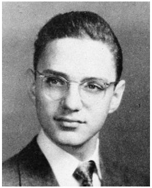
Past and future

If (T, F) is any log-integrable 2D cocycle with $\lambda_1 > \lambda_2$, the Oseledets direction $E_1(\omega)$ only depends on the past orbit $(T^n\omega)_{n<0}$, while $E_2(\omega)$ only depends on the future orbit $(T^n\omega)_{n\geq 0}$.

In particular, if the cocycle is one-step, then

$$E_1(\omega) = E_1(\omega^-), \quad \omega^- = (\dots, \omega_{-2}, \omega_{-1}),$$
$$E_2(\omega) = E_2(\omega^+), \quad \omega^+ = (\omega_0, \omega_1, \dots).$$

So the two directions E_1, E_2 are **independent**, i.e., their joint distribution is a product measure $m_1 \times m_2$, where m_1 and m_2 are measures on $\mathbb{R}P^1$ called *Furstenberg measures*.

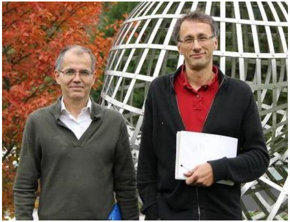


Summoning help

$$\int_{\Omega} |\log \sin \theta| \, d\mu = \int_{\mathbb{RP}^1} \underbrace{\int_{\mathbb{RP}^1} |\log \sin \measuredangle(\xi_1, \xi_2)| \, dm_1(\xi_1)}_{f(\xi_2)} \; dm_2(\xi_2) \, .$$

Theorem (Benoist–Quint, 2016)

If the matrix distribution ν has finite second moment and is strongly irreducible, then the function f above is continuous.



So our Theorem 2 (the criterion for log-integrability) holds true in the strongly irreducible case. ✓

Reduction to the reducible case

Suppose ν is *not* strongly irreducible. This means that there is a finite set $S \subset \mathbb{R}\mathrm{P}^1$ such that $g(S) = S$ for ν-a.e. $g \in \mathrm{GL}(2, \mathbb{R})$.

- If $\#S \geq 3$, then $\lambda_1(\nu) = \lambda_2(\nu)$: nothing to do. ✓
- If $\#S = 2$, then either $\lambda_1(\nu) = \lambda_2(\nu)$ or $\{E_1(\omega), E_2(\omega)\} = S$ for a.e. ω: nothing to do. ✓
- If $\#S = 1$, then the cocycle is *reducible*.

Reducible case

WLOG (after change of basis, rescaling, time reversal),

$$F(\omega) = \begin{pmatrix} a(\omega_0) & b(\omega_0) \\ 0 & 1 \end{pmatrix}, \quad \int \log|a| < 0\,.$$

Lemma (Oseledets data for triangular cocycle)

$$\lambda_1 = 0\,, \qquad \lambda_2 = \int \log|a|\,,$$

$$E_1(\omega) = \operatorname{span}\begin{pmatrix} X(\omega) \\ 1 \end{pmatrix}, \quad E_2(\omega) = \operatorname{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$X(\omega) = \sum_{n=0}^{\infty} a(T^{-1}\omega)a(T^{-2}\omega)\cdots a(T^{-n}\omega)b(T^{-n-1}\omega)\,.$$

Note: $\log\sin\measuredangle(E_1, E_2) \in L^1 \iff \log^+|X| \in L^1$

Proof of Theorem 1: an example with $\log \theta \notin L^1$

We leave Theorem 2 in limbo, and prove Theorem 1. Here's an explicit example:

$$F(\omega) := \begin{pmatrix} e^{-1} & e^{\psi(\omega_0)} \\ 0 & 1 \end{pmatrix}, \quad \text{where } \psi \in L^1 \setminus L^2, \ \psi \geq 0.$$

Proof: The previous formula becomes

$$X(\omega) = \sum_{n=0}^{\infty} e^{\psi(\omega_{-n-1})-n}.$$

WTS: $\boxed{\log X \notin L^1}$. (Note that $X \geq 1$, since $\psi \geq 0$.)

We use the bound $\boxed{X(\omega) \geq e^{Y(\omega)}}$, where

$$Y(\omega) := \sup_{n \geq 0} \Big[\psi(\omega_{-n-1}) - n \Big].$$

Lemma

Given $\psi \in L^1$, $\psi \geq 0$, *let*

$$Y(\omega) := \sup_{n \geq 0} \Big[\psi(\omega_{-n-1}) - n \Big].$$

Then $Y \in L^1 \iff \psi \in L^2$.

Proof: By the "layer cake formula",

$$\int Y \, d\mu < \infty \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \underbrace{\mu[Y \geq k]}_{b_k} < \infty \, .$$

$$\begin{aligned} 1 - b_k &= \mu[Y < k] \\ &= \mu\Big[\psi(\omega_{-n-1}) < n + k, \ \forall n \geq 0\Big] \\ &= \prod_{n=0}^{\infty} (1 - a_{n+k}) \text{ with } \boxed{a_j := \mu[\psi \geq j]} \quad \text{(independence).} \end{aligned}$$

Proof of the Lemma (continued): $a_j := \mu[\psi \geq j]$, with $\psi \in L^1$, so $0 \leq a_j \leq 1$ and $(a_j) \in \ell^1$.

$$b_k = \boxed{1 - \prod_{j=k}^{\infty}(1 - a_j) \asymp \sum_{j=k}^{\infty} a_j}. \qquad \text{Compare: } \left(\bigcap A_j^c\right)^c = \bigcup A_j$$

🤓

$$\begin{aligned} Y \in L^1 \quad &\Leftrightarrow \quad \sum_{k=1}^{\infty} b_k < \infty \\ &\Leftrightarrow \quad \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} a_j < \infty \\ &\Leftrightarrow \quad \sum_{j=1}^{\infty} j\, a_j < \infty \\ &\Leftrightarrow \quad \psi \in L^2 \end{aligned}$$

□ (Lemma and Thrm 1).

Proof of Theorem 2 (criterion for $\log \theta \in L^1$)

We are reduced to the case

$$F(\omega) := \begin{pmatrix} \pm e^{\phi(\omega_0)} & \pm e^{\psi(\omega_0)} \\ 0 & 1 \end{pmatrix}, \phi \in L^2, \ \psi^+ \in L^2, \ \int \phi < 0 .$$

WTS: $\boxed{\log^+ |X| \in L^1}$, where

$$X(\omega) = \sum_{n=0}^{\infty} \pm e^{S_n(\omega)}, \quad S_n(\omega) := \psi(\omega_{-n-1}) + \sum_{i=1}^{n} \phi(\omega_n) .$$

Proof of Theorem 2 (continued): Let $c > 0$.

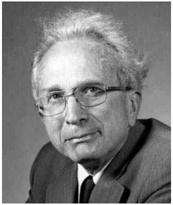
$$
\begin{aligned}
S_n(\omega) &\leq \psi^+(\omega_{-n-1}) + \sum_{i=1}^{n} \phi(\omega_n) \\
&= -cn + \left[\psi^+(\omega_{-n-1}) - cn\right] + \left[2cn + \sum_{i=1}^{n} \phi(\omega_{-i})\right] \\
&\leq -cn + Y(\omega) + Z(\omega),
\end{aligned}
$$

where Y and Z are defined by taking $\sup_{n \geq 0}[\cdots]$.

Previous Lemma applied to $c^{-1}\psi^+ \in L^2$ gives $\boxed{Y \in L^1}$.

Proof of Theorem 2 (continued): $c > 0$ small $\Rightarrow \int(2c+\phi) < 0 \Rightarrow$

$$\begin{aligned} Z(\omega) &:= \sup_{n\geq 0}\sum_{i=1}^{n}(2c+\phi(\omega_{-i})) \\ &= \text{rightmost position of a random walk with} \\ &\quad\ \text{drift to the left and square-integrable steps} \\ &\in L^1 \text{ by Kiefer and Wolfowitz (1956).} \end{aligned}$$



Proof of Theorem 2 (finale):

$$
\begin{aligned}
|X(\omega)| &\leq \sum_{n=0}^{\infty} e^{S_n(\omega)} \\
&\leq \sum_{n=0}^{\infty} e^{-cn+Y(\omega)+Z(\omega)} \\
&\leq \mathrm{Const}.e^{Y(\omega)+Z(\omega)}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\log^{+}|X(\omega)| &\leq \mathrm{Const} + Y(\omega) + Z(\omega) \\
&\in L^1 .
\end{aligned}
$$

$\square$

Flexibility for measurable cocycles: recap

Theorem

Given:

- *An ergodic automorphism T of non-atomic Lebesgue space $(\Omega, \mathcal{S}, \mu)$;*
- *a locally bounded function $N\colon \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R}$ s.t. $N(g) \geq \max\big(\|g\|, \|g^{-1}\|\big)$;*
- *numbers $r_1 > r_2$;*
- *a Borel probability measure η on $\mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^1 \smallsetminus \Delta$;*

Then there exists a measurable $F\colon \Omega \to \mathrm{GL}(2,\mathbb{R})$ with $\boxed{N \circ F \in L^1(\mu)}$ s.t. the cocycle (T, F) has Lyapunov exponents $\lambda_1 = r_1$, $\lambda_2 = r_2$, and the Oseledets spaces have joint distribution η – that is, $\boxed{(E_1, E_2)_* \mu = \eta}$.

Flexibility for measurable cocycles

Theorem (Simplified statement 👶)

Given:

- *An ergodic automorphism T of non-atomic Lebesgue space $(\Omega, \mathcal{S}, \mu)$;*
- *the function $N \colon \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R}$ given by $N(g) := \max\left(\|g\|, \|g^{-1}\|\right)$;*
- *numbers $r_1 > r_2$;*
- *a Borel probability measure η on $(0, \frac{\pi}{2}]$;*

Then there exists a measurable $F \colon \Omega \to \mathrm{GL}(2,\mathbb{R})$ with $\boxed{N \circ F \in L^1(\mu)}$ s.t. the cocycle (T, F) has Lyapunov exponents $\lambda_1 = r_1$, $\lambda_2 = r_2$, and the Oseledets angle $\theta = \measuredangle(E^1, E^2)$ has distribution η – that is, $\boxed{\theta_* \mu = \eta}$.

Nearly invariant functions with prescribed distributions

Lemma

Given:

- *An ergodic automorphism T of non-atomic Lebesgue space $(\Omega, \mathcal{S}, \mu)$;*
- *a Borel probability measure η on $\mathbb{R}$;*
- $\varepsilon > 0$.

Then there exists a measurable function $f : \Omega \to \mathbb{R}$ with

$$f_* \mu = \eta \quad \text{and} \quad \|f \circ T - f\|_{L^1(\mu)} < \varepsilon .$$

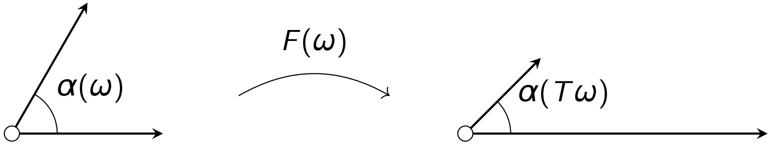
Remark

There is an L^∞ version of the Lemma (under a bounded gap assumption), which allows to prove Theorem 4 (Oseledets flexibility for bounded cocycles).

Lemma ⇒ (simplified) Theorem

The Lemma gives $\alpha\colon \Omega \to (0, \frac{\pi}{2}]$ with the prescribed distribution and $\|f \circ T - f\|_{L^1(\mu)} < \varepsilon$, where $f = \log \sin \frac{\alpha}{2}$.

Choice of the matrices:



Adjust stretch factors to get the desired Lyapunov exponents (no stretch when angles are too small). □

So we are left to prove the Lemma...

Skyscraper decomposition

Given an ergodic autom. $T\colon (\Omega,\mu) \to (\Omega,\mu)$ and a measble. set $B \subseteq \Omega$ with $\mu(B) > 0$, let B_k be the set of points in B whose first return to B occurs at time k. The *Kakutani skyscraper decomposition* is $\Omega = \bigsqcup_{k=1}^{\infty} \bigsqcup_{i=0}^{k-1} T^i(B_k) \bmod 0$.

$$
\begin{array}{cccc}
 & & & \cdots \\
 & & T^2B_3 & \cdots \\
 & & \uparrow & \\
 & TB_2 & TB_3 & \cdots \\
 & \uparrow & \uparrow & \\
B_1 & B_2 & B_3 & \cdots
\end{array}
\Big\} \; B
$$

Kac's lemma

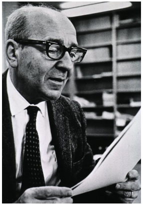
The measure of the k^{th} tower is:

$$\pi_k = k\mu(B_k)$$

In particular,

$$\sum_{k=1}^{\infty} \pi_k = 1$$

Equivalently, the average return time to B is $\frac{1}{\mu(B)}$ (Kac's lemma).



A converse to Kac's lemma: flexibility of π

Theorem (Alpern–Prasad, 1990)

If T is an ergodic automorphism of a non-atomic Lebesgue space and $\pi = (\pi_1, \pi_2, \dots)$ is s.t.

$$\pi_k \geq 0, \; \sum_{k=1}^{\infty} \pi_k = 1, \; \mathrm{GCD}\{k\,;\, \pi_k \neq 0\} = 1,$$

there exists a skyscraper whose towers have measures as specified by the sequence π.

Example (Rokhlin–Halmos Lemma, 1940s)

$$\pi_k := \begin{cases} \varepsilon & \text{if } k = 1, \\ 1 - \varepsilon & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma (Nearly invariant f with prescribed distrib.)

Given the automorphism T, a Borel probability measure η on $\mathbb{R}$, and $\varepsilon > 0$, there exists a measurable function $f : \Omega \to \mathbb{R}$ with $f_*\mu = \eta$ and $\|f \circ T - f\|_{L^1(\mu)} < \varepsilon$.

Proof:

- Decompose η as a convex combination $\sum p_n \eta_n$ of compactly supported measures.
- Take a sequence $k_n \nearrow \infty$ fast, with GCD $= 1$.
- By Alpern–Prasad, there is a skyscraper composed of towers of heights $k_1, k_2, \ldots$ and corresponding masses $p_1, p_2, \ldots$
- Choose a function f which is invariant on the tower of height k_n and has distribution η_n.
- Estimate the L^1 norm... $\square$