A PROOF OF DENJOY THEOREM

Definition. Let (X, Y) be a pair of finite subsets of the circle $\mathbb{T} \coloneqq \mathbb{R}/Z$. A *matching* for this pair is a bijection $\phi \colon X \to Y$ such that each $x \in X$ is a *neighbor* of $y = \phi(x)$ in the sense that there exists an interval $I \subset \mathbb{T}$ such that $I \cap (X \cup Y) = \{x, y\}$.

Lemma 1. *Let $f \in \mathrm{Homeo}_+(\mathbb{T})$ have irrational rotation number α. Let p/q be a Dirichlet approximation for α.*[1] *Then any pair of orbit segments of length q admits a matching.*

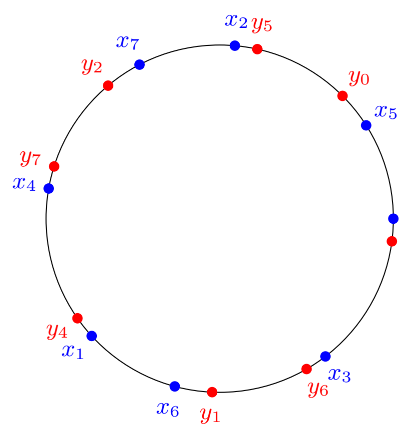

FIGURE 1. Lemma 1 with $f = R_\alpha$, $\alpha = (\sqrt{5} - 1)/2$, and $q = 8$.

Proof. Consider any pair of orbit segments of length q:

$$X = \left\{x_0, x_1 = f(x_0), \ldots, x_{q-1} = f^{q-1}(x_0)\right\},$$
$$Y = \left\{y_0, y_1 = f(y_0), \ldots, y_{q-1} = f^{q-1}(y_0)\right\}.$$

Let us deal first with the case where $X \cap Y \neq \emptyset$ and f is the rotation R_α. Interchanging X and Y if necessary, we can assume that $x_0 = y_k$ for some k. Let us check that the map $\phi_k \colon x_i \mapsto y_{i+k \bmod q}$ is a matching. Indeed:

- If $0 \le i < q - k$ then x_i and y_{i+k} are neighbors because they are equal.
- If $q - k \le i < q$ then the shortest interval from $x_i = f^{i+k}(y_0)$ to y_{i+k-q} has length equal to the distance between $f^q(y_0)$ and y_0. This distance is less than the minimal separation between points of any orbit segment of length q, and so the interior of the interval does not intersect any points from $X \cup Y$. Hence x_i and y_{i+k-q} are neighbors.

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[1] Equivalently, q is a closest return time for the irrational rotation R_α.

If f is not a rotation, still the cyclic order of any orbit segment is the same as for the rotation R_α, and therefore the assumption $X \cap Y \neq \emptyset$ guarantees the existence of a matching.

Now consider the case where the two segments X and Y are disjoint. Let us deform continuously the second orbit segment: let $Y^{(t)} = \{y_0^{(t)}, \dots, y_{q-1}^{(t)}\}$ where $y_i^{(t)} := f^i(y_0+t)$. Let τ be the least positive number for which $Y^{(\tau)}$ intersects X. As shown above, there is a matching between X and $Y^{(\tau)}$ of the form $x_i \mapsto y_{i+k \bmod q}^{(\tau)}$. Now, if t is slightly smaller than τ then $x_i \mapsto y_{i+k \bmod q}^{(t)}$ is a matching between X and $Y^{(t)}$ – matchings are clearly stable under perturbations. As t decreases to 0, no new crossings arise, and therefore the same formula defines a matching. □

Lemma 2 (Bounded distortion)**.** *Let $f \in \mathrm{Diff}^2_+(\mathbb{T})$ have irrational rotation number α. Then there exists $\lambda > 0$ such that whenever p/q is Dirichlet approximation for α, we have:*

$$\lambda^{-1} \leq (f^q)' \leq \lambda\,.$$

Proof. Let p/q be a Dirichlet approximation for α. Take any two points x, $y \in \mathbb{T}$, and write $x_i := f^i(x)$, $y_i := f^i(y)$, for $0 \leq i < q$. By Lemma 1, there is a matching between these two segments of orbit. This means that there exists a permutation σ of $\{0, 1, \dots, q-1\}$ and there exist q pairwise disjoint (possibly degenerate) intervals I_i such that $\partial I_i = \{x_i, y_{\sigma(i)}\}$. Therefore:

$$\begin{aligned} |\log(f^q)'(x) - \log(f^q)'(y)| &= \left| \sum_{i=0}^{q-1} \log f'(x_i) - \log f'(y_i) \right| \\ &\leq \sum_{i=0}^{q-1} \left| \log f'(x_i) - \log f'(y_{\sigma(i)}) \right| \\ &= \sum_{i=0}^{q-1} \left| \int_{I_i} (\log f')' \right| \\ &\leq \int_{\mathbb{T}} |(\log f')'|\,. \end{aligned}$$

That is, $\lambda^{-1} \leq (f^q)'(x)/(f^q)'(y) \leq \lambda$ where $\lambda := \int_{\mathbb{T}} |f''|/f'$. Since $\int_{\mathbb{T}} \log(f^q)' = 1$, there exists y such that $(f^q)'(y) = 1$, and the lemma follows. □

Theorem (Denjoy)**.** *Let $f \in \mathrm{Diff}^2_+(\mathbb{T})$ have irrational rotation number α. Then f is conjugate to the irrational rotation R_α.*

Proof. By Poincaré Classification Theorem, it is sufficient to show that f admits no wandering interval, i.e., an interval that is disjoint from all its future and past iterates.

For any interval $I \subset \mathbb{T}$, by Lemma 2 we have $|f^q(I)| \geq \lambda^{-1}|I|$ whenever q is the denominator of a Dirichlet approximation for α, where $\lambda \geq 1$ is a finite constant. Since there are infinitely many such q's, we conclude that I cannot be wandering. □

References

[1] Katok, A.; Hasselblatt, B. – *Introduction to the modern theory of dynamical systems.* Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[2] Khanin, K. – *Ergodic theory, continued fractions and renormalizations.* Chapter 3 of *XIV Escuela Latinoamericana de Matemáticas*, Solś, Uruguay, 2005.

[3] Milnor, J. – *Introductory dynamics lectures.* www.math.stonybrook.edu/∼jack/DYNOTES

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