AN ELEMENTARY PROBLEM ON PRODUCTS OF TWO 2×2 MATRICES

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Let $H \in \mathrm{SL}(2,\mathbb{R})$ be a hyperbolic matrix, that is, whose eigenvalues satisfy $|\lambda_1| > 1 > |\lambda_2| = |\lambda_1|^{-1}$. Then norms of the powers of H grow exponentially fast.

A more interesting situation is to consider products of matrices $A_1 \cdots A_n$ where most A_i's are equal to H, but a few of them are equal to another matrix R, say, a rotation. These matrices R can be considered as "impurities" in the product $A_1 \dots A_n$. We may ask: *If impurities are "rare", can we recover uniform exponential growth for the products?* Let us be more precise:

Definition. *Let H, $R \in \mathrm{SL}(2,\mathbb{R})$, with H hyperbolic and R elliptic (i.e., conjugate to a rotation). We say that* $\boxed{H \text{ resists } R \text{ impurities}}$ *if if there exist:*

- *(a maximum impurity rate) $\varepsilon > 0$, and*
- *(a minimum "Lyapunov exponent") $\lambda > 0$*

such that for any word w in the letters H and R containing at most $\varepsilon|w|$ letters R (where $|w|$ is the length of the word), the corresponding matrix product P_w has norm

$$\|P_w\| > e^{\lambda|w|}.$$

Nonexample. *Let*

$$H = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \qquad R = R_{\pi/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then H does not resist R impurities. Indeed, there are words on H and R that contain an arbitrarily small proportion of R's and have small product:

$$\|H^n R H^n\| = 1 \quad \forall n \geq 1.$$

We conjecture that the phenomenon illustrated by the nonexample is atypical:

Conjecture (J.B. & Bassam Fayad, Bull. Braz. Math. Soc. 2006)**.** *The set of "resistant pairs"*

$$\mathcal{R} := \big\{(H,R) \in \textit{Hyperbolic} \times \textit{Elliptic};\ \textit{H resists R impurities}\big\}$$

has full Lebesgue measure.

Remark (Weaker form of the conjecture)**.** *It is not even known if the set $\mathcal{R}$ has positive Lebesgue measure.*

The difficulty of this type of problem arises from the fact that impurity resistance involves *all* words obeying the frequency condition, and not just "typical" words. Indeed, the classic result of Furstenberg (1963) on random i.i.d. matrix products says that under explicit very mild conditions on H, R, the typical products grow exponentially with length, i.e., have positive Lyapunov exponent. But this is much weaker than what we ask.

Some remarks:

Remark (B. & Fayad)**.** *The set $\mathcal{R}$ is meager in the sense of Baire.*

Proof. "Fragile pairs" trivially form a G_δ set. Denseness is proved imitating the nonexemple above. □

Remark (C. Bonatti)**.** $\mathcal{R} \neq \varnothing$.

Proof. Take (H,R) such that $R = R_{\pi/2}$ and the pair $(H, \tilde{H} = R^{-1}HR)$ has a common strictly invariant bi-cone. □

Remark. *Bochi–Fayad conjecture holds if instead of allowing εn letters R's in each word of length n, we allow only:*

- $\boxed{\varepsilon n^\alpha}$ *letters R, for any fixed* $0 < \alpha < 1/2$ *[B. Fayad & R. Krikorian, Nonlinearity 2008]; or even better*
- $\boxed{\dfrac{\varepsilon n}{\log n \log\log n}}$ *letters R [A. Avila & T. Roblin, JMD 2009]*

Remark. *Some positive results were obtained in the simpler situation* $H = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$. *[Allen, Seeger & Unger, Involve 2011]*