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Lyapunov exponents.

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Trieste, 07/07/2008. (1)

Furstemberg (Independent Random Products)

$\tilde{\mu}$ in $SL(2,\mathbb{R})$, $\int \log \|x\| \, d\mu(x) < +\infty$ (integrability)

$A_1, \dots, A_n, \dots$

$\Sigma = SL(2,\mathbb{R})^{\mathbb{Z}}$, $\mu = \tilde{\mu}^{\mathbb{Z}}$ on Σ

$$f : \Sigma \longrightarrow \Sigma \quad \text{shift}$$
$$(x_n) \longmapsto (x_{n+1})_{n\in\mathbb{Z}}$$

Think of $\tilde{\mu}$ as finite support

$\boxed{\lambda = 0}$ Three cases:

1) $\operatorname{supp}\tilde{\mu} \subset$ compact subgroup of $SL(2,\mathbb{R})$
2) $\operatorname{supp}\tilde{\mu} \subset$ triangular subgroup

$\begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix}$ **Obs**: $\exists\, w \in \mathbb{P}\mathbb{R}^2$ s.t. $x \cdot w = w \quad \forall x \in \operatorname{supp}\tilde{\mu}$

$$\lambda = \left| \int \ln \frac{\|x \cdot w\|}{\|w\|} \, d\mu(x) \right.$$

3) $\exists\, \{w_1, w_2\} \subset \mathbb{P}\mathbb{R}^2$ s.t.

$\forall x \in \operatorname{supp}\tilde{\mu}$, $x \cdot \{w_1, w_2\} = \{w_1, w_2\}$

②

<u>Furstenberg Theorem</u>: In all other cases, we have $\lambda > 0$

(Ledrappier, Raugi-Guivarch, Bonatti-G-M-Viana, Bonatti-Viana)

<u>Exercise</u>: Formulate differently (Furstenberg Thm) in terms of <u>twisting</u> ($\forall w \in \mathbb{P}\mathbb{R}^2$, $\{w_1, \dots, w_k\} \in \mathbb{P}\mathbb{R}^2$. $\exists\, x_n, \dots, x_1 \in \operatorname{supp} \tilde{\mu}$, $X = x_n \cdot \ldots \cdot x_1$, $X \cdot w_i \neq w_i$, $\forall\, 1 \leq i \leq k$) and <u>pinching</u> ($\exists\, X = x_n \cdot \ldots \cdot x_1$ with arbitrarily large norm).

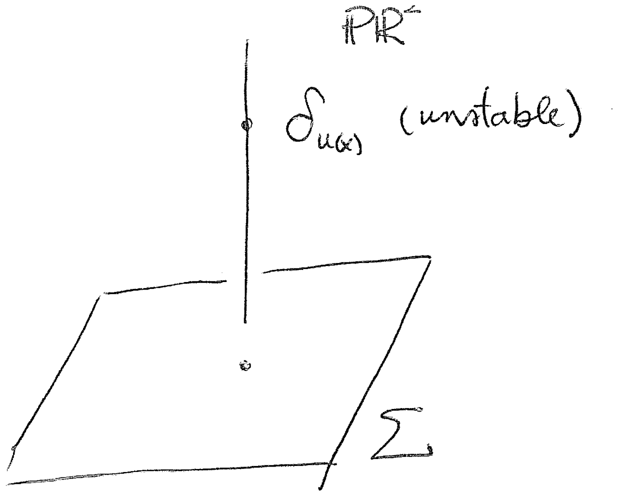
Suppose $\lambda > 0$, almost every x, $\exists\, u(x), s(x) \in \mathbb{P}\mathbb{R}^2$.

<u>Projective Action</u>

$$F : \Sigma \times \mathbb{P}\mathbb{R}^2 \longrightarrow \Sigma \times \mathbb{P}\mathbb{R}^2 \quad \text{where } A((x_n)) = x_0$$

$$(x, w) \longmapsto A(x) \cdot w = F(x, w)$$

Define an invariant measure ν^u, on each fiber over x.



$$X \subset \Sigma \times \mathbb{P}\mathbb{R}^2$$

$$\nu^u(X) = \mu\,(x \in \Sigma : (x, u(x)) \in X)$$

ν^u is invariant

$$\varphi(x,\omega) = \ln \frac{\| A(x)\cdot \omega \|}{\|\omega\|}$$

$$\lambda = \int \varphi \, d\nu^u$$

$$\frac{1}{n} \ln \frac{\| A^n(x) \cdot u(x) \|}{\| u(x) \|} = \frac{1}{n} \sum_{k=0}^{n-1} \varphi\,(F^k(x, u(x)))$$

$$\forall \omega \in \mathbb{T}^1(x) \setminus \{s(x)\}$$

$$\operatorname{dist}\,(A^n(x)\cdot \omega,\ u(f^n(x))) = 0$$

$$F^n_*\,(\mu \times \mathrm{Leb}) \longrightarrow \nu^u$$

ν^u only depends on the past.

$$\Sigma = \Sigma^- \times \Sigma^+$$

$$\Sigma^- = SL(2,\mathbb{R})^{\mathbb{Z}^-}, \quad \Sigma^+ = SL(2,\mathbb{R})^{\mathbb{N}}$$

$$\nu^u(\overset{\Sigma^-}{X} \times \overset{\Sigma^+}{Y} \times \overset{P\mathbb{R}^2}{Z}) = \nu^u(X \times \Sigma^+ \times Z)\,\mu^+(Y)$$

measure projected on $\Sigma \times P\mathbb{R}^2$

__Goal__: Construct ν^u.

__DEF__: ν an invariant measure on $\Sigma \times P\mathbb{R}^2$ is called an __u-state__ if only if depends on the past.

(1) Construct an u-state ν.

(2) Show that conditional measures ν_x are Diracs (attraction).

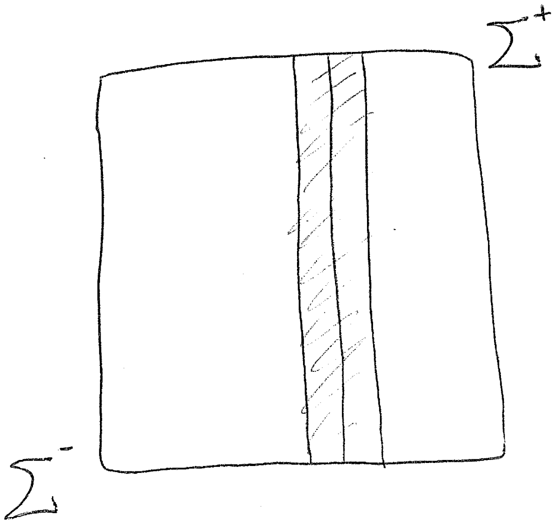
Start with $\tilde{\nu}$ (for instance, Leb) that only depends on the past and take Cesaro limits:

$$\frac{1}{n} \sum_{k=0}^{n-1} F_*^k \tilde{\nu}$$

The limit is u-state.

ν_x depends measurably on x.

Measurable $\Longrightarrow$ continuity properties a.e.



$\tilde{x}$ μ-typical point

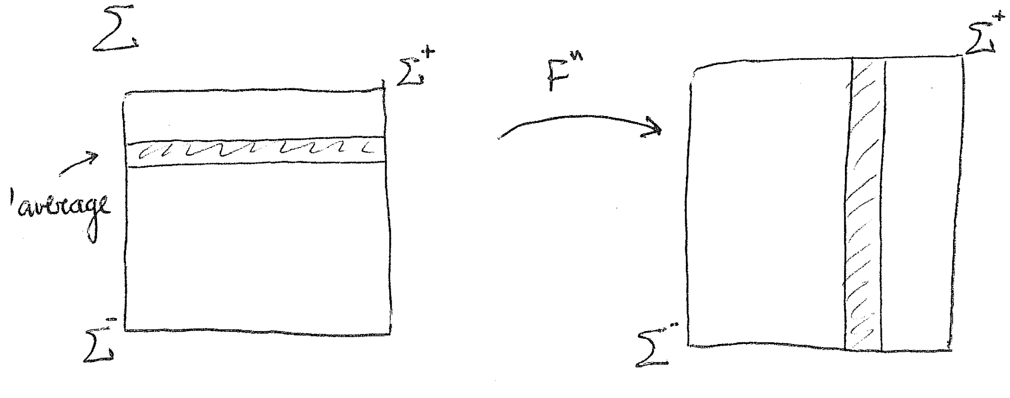
for most x close to $\tilde{x}$

$\nu_{\tilde{x}}$ close to ν_x

$$\frac{1}{\mu([\tilde{x}_{-n},\dots,\tilde{x}_{-1}])} \int \nu_x \, d\mu(x) \longrightarrow \nu_{\tilde{x}}$$

x with same past as $\tilde{x}$ up to order n

$$\eta = \int_{\Sigma} \nu_x \, d\mu(x)$$, η only depends on ν



$A^n(f^n(\tilde{x}))\cdot\eta \to \nu_{\tilde{x}}$

η is a stationary measure

$$Z \subset \mathbb{P}\mathbb{R}^2, \quad \eta(Z) = \int_{SL(2,\mathbb{R})} \eta(x^{-1}(Z))\, d\tilde{\mu}(x)$$

LEMMA: Twisting $\Longrightarrow$ η is non-atomic.

Proof: Suppose there are atoms. Consider atoms of maximal weight $\{w_1, \dots, w_k\}$

$$\eta(w_i) = \int \eta(x^{-1}(w_i))\, d\tilde{\mu}(x)$$

$$\eta(x^{-1}(w_i)) \leq \eta(w_i)$$

$$\eta(w_i) = \eta(x^{-1}(w_i)) \quad \forall x \in \operatorname{supp} \tilde{\mu}, \ \forall i$$

$$\Downarrow$$

$$x^{-1}\{w_1, \dots, w_k\} = \{w_1, \dots, w_k\}$$

#

LEMMA: Pinching + twisting $\Longrightarrow$ $A^n(f^{-n}(x))_* \eta \to \nu_x$ (Dirac) along a subsequence for a positive measure of x.

Trieste, 08/07/2008. (Artur Avila, continuation ...)

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Furstenberg Theorem

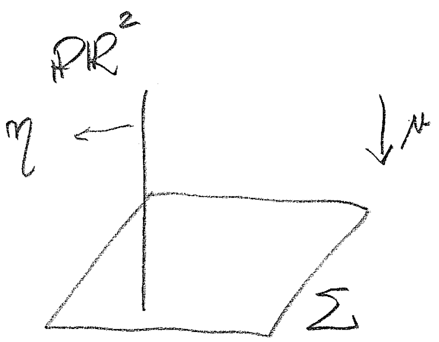
Pinching and Twisting $\Longrightarrow \lambda > 0$

Independent Random Products in $SL(2,\mathbb{R})$

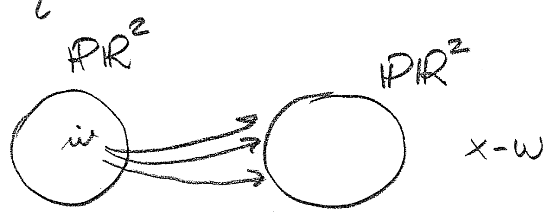
two sided shift

Constructed u-state ν on $\Sigma \times \mathbb{P}\mathbb{R}^2$

ν invariant $\quad F = \Sigma \times \mathbb{P}\mathbb{R}^2 \longrightarrow \Sigma \times \mathbb{P}\mathbb{R}^2$



η stationary measure



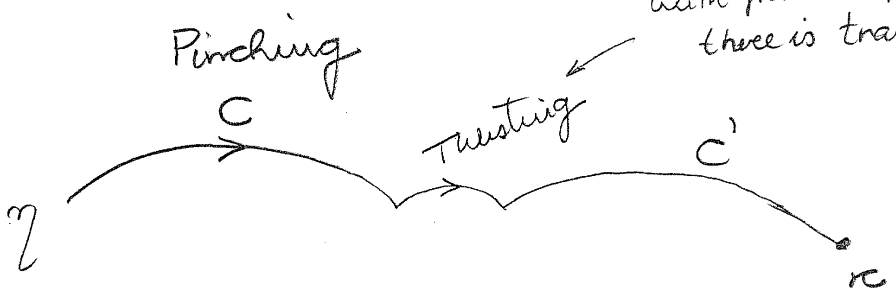
ν_x, $x \in \Sigma$ is conditional measure

LEMMA 1: $A^n(f^{-n}(x))_* \eta \longrightarrow \nu_x$ for a.e. x

LEMMA 2: Twisting $\Longrightarrow$ η has no atoms.

LEMMA 3: Pinching + Twisting $\Longrightarrow$ $A^n(f^{-n}(x)) \cdot \eta \to$ Dirac along subsequence for positive measure of x.

with positive probability there is transversality



1) $\|C\|$ has large norm

$C = R D R'$

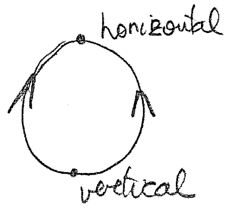
rotations

diagonal $\begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix}$ K is large

$C_* \eta$ is concentrated near some direction

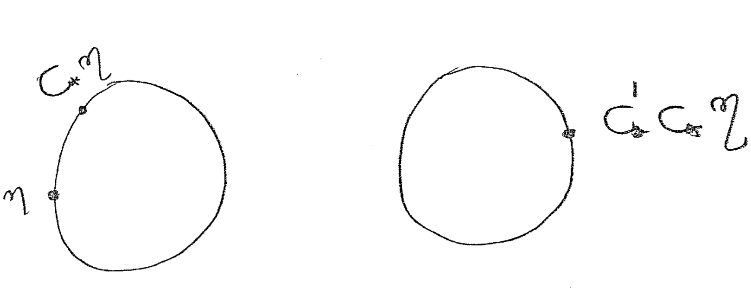
non atomic

$\mathbb{P}\mathbb{R}^2$



2) C' may have bounded norm
C' may be large

(Twisting : $\forall \omega, \omega'$ can find B which puts $B\omega \pitchfork \omega'$)



~~#~~ Lemma 3

ν u-state (depends only on the past)

$$\nu_x = \delta_{u(x)}$$

$$\| A^n(f^{-n}(x)) \| \longrightarrow +\infty$$

(since it concentrates η into a Dirac)

ν' s-state (depends only on future)

$$\nu'_x = \delta_{s(x)}$$

LEMMA 4: $u(x) \neq s(x)$ for a.e. x.

proof. If not $u(x) = s(x)$ for a.e. x

only depends on past

only depends on future

$u(x)$ is constant

η is Dirac

$Z \subset \Sigma$ such that $d(u(x), s(x)) > \varepsilon$.

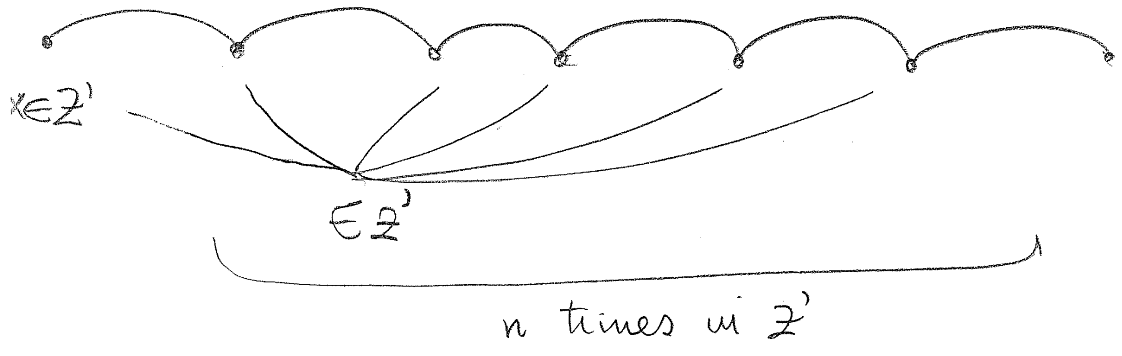
$$\lim \frac{\| A^n(x)\, u(x) \|}{\| u(x) \|} = \infty$$ for the subsequence of n's with $f^n(x) \in Z$

(EXERCISE)

The most expanded direction ω of $A^n(f^{-n}(x))$ is such that $A^n(f^{-n}(x)) \cdot \omega \longrightarrow u(x)$. 10

$Z' \subset X$. If $x \in Z'$, $f^n(x) \in Z'$, $n > 0$

$$\ln \frac{\| A^n(x) \cdot u(x) \|}{\| u(x) \|} > 1$$



$$\frac{\| A^N(x) \cdot u(x) \|}{\| u(x) \|} \gtrsim e^n \sim e^{N\mu(Z')} \Longrightarrow \lambda \geq \mu(Z') > 0$$

$$\frac{n}{N} \sim \mu(Z')$$

Furstenberg

Generalization of Furstenberg

- $\lambda = 0 \Rightarrow$ strong constraints

PROBLEM

(BOCHI) $f : X \longrightarrow X$ homeomorphism, for generic $A : X \longrightarrow SL(2,\mathbb{R})$ continuous, μ on X with full support, either $\lambda = 0$ or (f, A) is uniformly hyperbolic.

Example: $f : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$
$x \longmapsto x + \alpha$

$A : \mathbb{R}/\mathbb{Z} \longrightarrow SL(2,\mathbb{R})$ not homotopic a constant

Then (f, A) is not uniformly hyperbolic.

THEOREM (A., BOCHI): $\forall A$, $\displaystyle\int_{\mathbb{R}/\mathbb{Z}} \lambda(R_\theta A)\, d\theta = \int_X \ln \frac{\|A\| + \|A\|^{-1}}{2}\, d\mu$

(if 0 then $\|A\| = 1$ $A(x) \in SO(2,\mathbb{R})$)

$R_\theta A(x) = R_\theta(A(x))$

THEOREM (A., DAMANIK) For generic A, for a.e. θ

$$\lambda(R_\theta A) > 0.$$

- $SL(2,\mathbb{R})$ cocycle is always close to (complex perturbation) uniform. hyperbolic $SL(2,\mathbb{C})$ cocycle.

LEMMA 5: $(f, R_\theta A)$ is U.H. $\forall\theta$ with $\mathrm{Im}\,\theta > 0$.

$$R_\theta := \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}, \qquad \|(R_\theta A)^n(x)\| \geq C(1+\delta)^n$$

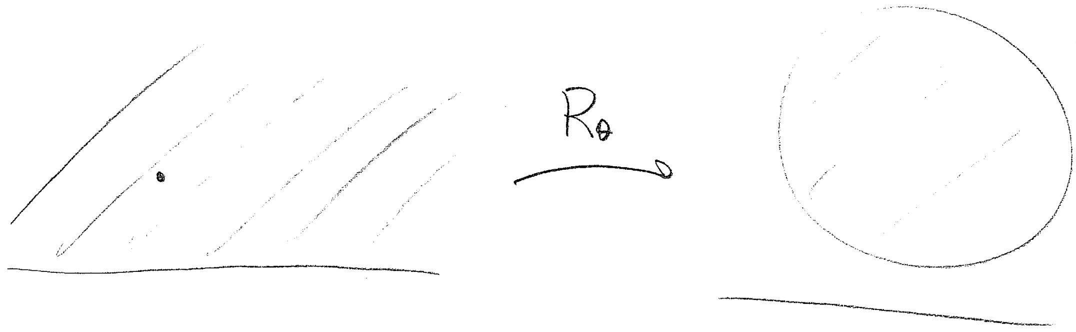
Proof:

$$\mathbb{P}\mathbb{C}^2 \sim \overline{\mathbb{C}} = \mathbb{C} \cup \{+\infty\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

$SL(2,\mathbb{R})$ fixes $\mathbb{R}$, $\mathbb{H}$ (hyperb. plane), $\mathrm{Im}(\theta) > 0$

$R_\theta \cdot \mathbb{H}$ is compactly contained in $\mathbb{H}$.



$$\mathbb{H} \xrightarrow{A} \mathbb{H} \xrightarrow{R_\theta} \mathbb{H} \to \cdots$$

Schwarz Lemma

Poincaré metric on $\mathbb{H}$ preserved under A

strictly contracted by R_θ.

$(R_\theta A)^n \leftarrow$ exponential contraction

$\omega = i$

$\omega' = 1 + i$

uniformly exponential Asymptotic

$\Downarrow$

$\|(R_\theta A)^n\|$ grows uniformly expon. fast

Example :

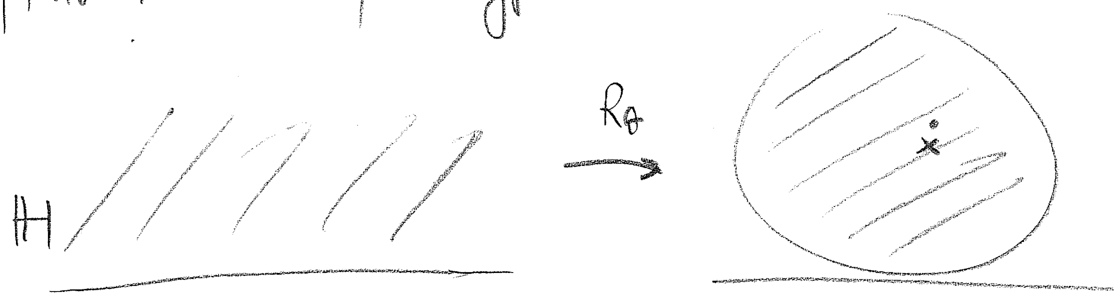
1) Shift dynamics has lots of periodic orbits

If the cocycle restricted to periodic orbit is not elliptic then the Lyapunov expon. is positive.
(λ is not vanishing in a neighborhood)

Approach :

$F_\theta : X \times \bar{\mathbb{C}} \to X \times \bar{\mathbb{C}}$, $\mathrm{Im}\,\theta > 0$ then

$(f, R_\theta A)$ is unif. hyperb.



$UH \implies \exists\, u(\theta, x)$ (unstable direction)
$s(\theta, x)$

$\mathbb{H} = \{\mathrm{Im}\, z > 0\}$ and $\mathbb{H}^- = \{\mathrm{Im}\, z < 0\}$

$u(\theta, x) \in R_\theta \cdot \mathbb{H} \subset \mathbb{H}$

$s(\theta, x) \in R_\theta^{-1}\, \mathbb{H}^- \subset \mathbb{H}^-$

For x fixed, $\theta \longmapsto u(\theta, x)$ is a holomorphic function
$\mathbb{H} \longrightarrow \mathbb{H}$

Trieste, 09/07/2008.

KOTANI-LIKE Theorems

$f: X \to X$ homeom. , $A: X \to SL(2,\mathbb{R})$ continuous

μ is a measure with full support

$R_\theta A$ one-parameter

<u>Theorem 1</u>: If $\lambda = 0$ for a positive measure set of θ then "the future determines the past" (modulo rotatio

Invertible cocycle

$\{A(f^n(x))\}_{n\in\mathbb{Z}}$ mod $SO(2,\mathbb{R})$, Knowledge of $(A(f^n(x)))_{n\geq 0}$ mod $SO(2,\mathbb{R})$

$\Downarrow$

Knowledge of $(A(f^n(x)))_{n\in\mathbb{Z}}$ mod $SO(2,\mathbb{R})$

x and y have the <u>same</u> <u>future</u>:

$$(A(f^n(x)))_{n\geq 0} = (A(f^n(y)))_{n\geq 0}$$

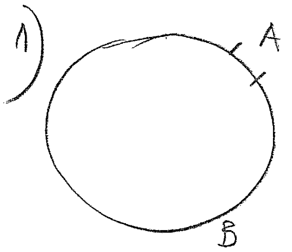
$$A \in SL(2,\mathbb{R}) / SO(2,\mathbb{R})$$

<u>Theorem 2</u>: If A is finite valued , $\#\{A(x)\}_{x\in X} < +\infty$, non-periodic (modulo $SO(2,\mathbb{R})$) ($\exists x \in X$, $(A(f^n(x)))_{n\in\mathbb{Z}}$ is

not periodic) then $\lambda > 0$ for a.e. θ.

Examples:

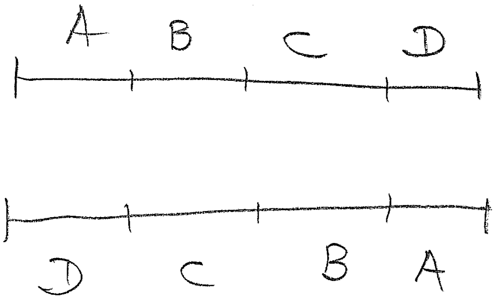
1)



$x \longmapsto x + \alpha$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$

2) Interval Exchang Transf.



$f : X \longrightarrow X$ invertible measureable

$A : X \longrightarrow SL(2, \mathbb{R}) \qquad A \in L^{\infty}$

$\Pi : X \longrightarrow SL(2, \mathbb{R})^{\mathbb{Z}}$

$x \longmapsto (A(f^n(x)))_{n \in \mathbb{Z}}$

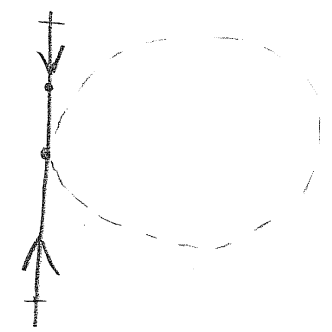
$\Pi_* \mu$ is a measure on $SL(2, \mathbb{R})^{\mathbb{Z}}$ which is invariant under the shift.

$\tilde{f} : \operatorname{supp} \Pi_* \mu \longrightarrow \operatorname{supp} \Pi_* \mu$ (restriction of shift)

$\tilde{A}(x) = x_0$

Proof (Theorem 2):

μ is a measure on shift with finitely many symbols.



not the same entire history.

Hyp $\Longrightarrow$ supp μ contains two distinct points in the same stable manifold

same future

Theorem 3: I open interval, $\lambda(R_\theta A) = 0$ for $\theta \in I$.

$\exists\, B : I \times X \longrightarrow SL(2,\mathbb{R})$:

1) $\left(B(\theta, f(x)) \cdot R_\theta A(x)\, B(\theta, x)^{-1} \right) \in SO(2,\mathbb{R}) \quad \forall \theta, x$

2) B is continuous in θ and x

3) B is analytic in θ.

It can be basis for a search of density of positivity of λ

$$u(\theta, x) = \lim_{n \to +\infty} \underbrace{(R_\theta A)^n (f^{-n}(x)) \cdot i}_{\substack{\text{holomorphic function of } \theta \\ \text{taking values in } \mathbb{H}}}$$

(Montel Theorem $\Longrightarrow$ limit is holomorphic)

Consider $u(\theta, x)$, $\operatorname{Im} \theta > 0$, $x \in X$

We want to consider limits with θ real.

<u>LEMMA</u> (FATOU) Let $\varphi : \mathbb{H} \to \mathbb{H}$ be holomorphic for a.e. $\sigma \in \mathbb{R}$, $\lim_{t \to 0_+} \varphi(\sigma + it)$ exists. ("Existence of non-tangential limits")

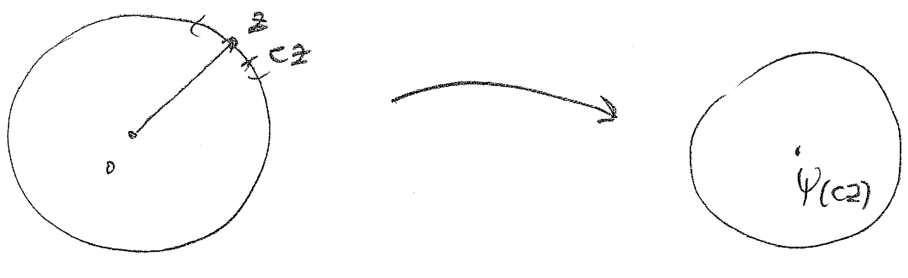
<u>proof (Idea)</u>: Can work on $\mathbb{D}$ (disc)

$\varphi : \mathbb{D} \to \mathbb{D}$ is a bounded holom. function.

Poisson Kernel

$$\varphi(z) = \int_0^1 P(z, e^{2\pi i \rho})\, \tilde{\varphi}(e^{2\pi i \rho})\, d\rho \quad \text{for } \tilde{\varphi} \in L^\infty(S^1)$$

$c \sim 1$, apply $\psi : \mathbb{D} \to \mathbb{D}$, $\psi(cz) = 0$



Complex Analysis

For every x; a.e. $\theta \in \mathbb{R}$
Can define $u(x,\theta)$ as non-tangential limit

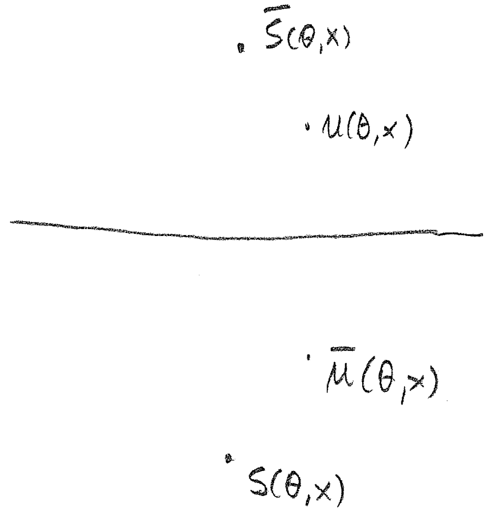
for a.e. θ, for a.e. x, $u(\theta,x)$ is defined.

Similar for $s(\theta,x)$

$$\begin{cases} R_\theta A \cdot u(\theta,x) = u(\theta, f(x)) \\ R_\theta A \; s(\theta,x) = s(\theta, f(x)) \end{cases}$$

If $\lambda > 0$ (at θ) then u and s must be the Oseledets directions. In particular, u and s are real.

<u>Theorem 4</u> (Approximation) For a.e. θ with $\lambda = 0$,
$u(\theta,x) \in \mathbb{H}$ for a.e. x.



* a $SL(2,\mathbb{R})$ matrix that fixes four directions is $\pm Id$

* $(f, R_\theta A)$ is conjugate to a cocycle with values in $\{\pm Id\}$ (unlikely to happen)

Generic Situation: $\bar{u}(\theta,x) = s(\theta,x)$.

$u(\theta, x)$ only depends on the past

$S(\theta, x)$ " " on the future

- Sequence of matrices $\longrightarrow u$

Holomorphic function

$u \xrightarrow{\text{determine}}$ sequence.

Trieste, 10/07/2008.

$\lambda = 0 \Longrightarrow$ determinism

is positive measure

$\lambda = 0 \Longrightarrow u$ and s functions are not real valued $\Longrightarrow u = \bar{s}$

<u>Theorem</u>: $\lambda = 0$ on I (open interval), $B : I \times X \to SL(2,\mathbb{R})$

$$B(\theta, f(x))\, R_\theta\, A(x)\, B(\theta, x)^{-1} \in SO(2,\mathbb{R})$$

1) B continuous in θ, x

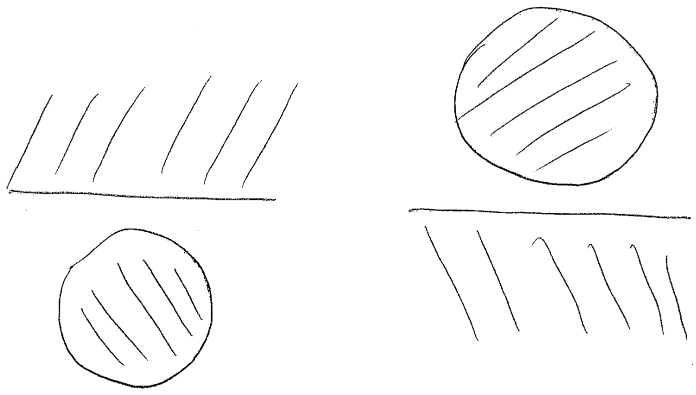
2) B analytic in θ

$$u, s : \mathbb{H} \longrightarrow \mathbb{H}$$

$$\lim_{t \to 0_+} u(\theta + it) = \lim_{t \to 0_+} \overline{s(\theta + it)}$$

$u(z) = \overline{s(\bar{z})}$

$\mathrm{Im}\, z < 0$



Function $u : \mathbb{H} \cup \mathbb{H}^- \longrightarrow \mathbb{H}$

At I the functions glue a.e. $\overset{\text{exercise}}{\Longrightarrow}$ u extends analytically though I

(case done in class if continuous gluing)

(general case by convolution)

- dependence on θ is analytic.
- dependence on x is continuos

$$u : \mathbb{C} \setminus (\mathbb{R} \setminus I) \longrightarrow \mathbb{H} \sim \mathbb{D} \quad \text{equicontinuity, holomorphic}$$

$$u : \mathbb{C} \setminus (\mathbb{R} \setminus I) \times X \longrightarrow \mathbb{H}$$

For $\theta \in I$, choose $B \in SL(2,\mathbb{R})$.

$$B(\theta, x) \cdot u(\theta, x) = i$$

$$R \in SL(2,\mathbb{R}), \quad Ri = i \iff R \in SO(2,\mathbb{R})$$

$$\overbrace{B(\theta, f(x)) \cdot R_\theta A(x) \cdot B(\theta,x)^{-1}}^{\in SO(2,\mathbb{R})} \cdot i = i$$

$$\underbrace{R_\theta A(x) \cdot \underbrace{B(\theta,x)^{-1} \cdot i}_{u(\theta,x)}}_{u(\theta, f(x))}$$

Theorem: For a.e. θ with $\lambda = 0$

$$\underbrace{\int \frac{1+|u|^2}{2\,\mathrm{Im}\,u}\,du \quad < +\infty}$$

$$\exists B : X \longrightarrow SL(2,\mathbb{R})$$

$$B(\theta, f(x)) \cdot R_\theta A(x) \; B(\theta,x)^{-1} \longleftarrow \text{in } SO(2,\mathbb{R})$$

$$\|B\|_{HS}^2 = \frac{1+|u|^2}{2\,\mathrm{Im}\,u}, \quad \int \|B\|_{HS}^2 \, d\mu < +\infty$$

$R_\theta A$ is L^2- conjugated to rotations

$\lambda = 0$

$$\frac{1}{n}\int \ln (R_\theta A)^n \, d\mu \longrightarrow 0 \quad \text{(always)}$$

$$(R_\theta A)^n = B(\theta, f^n(x)) \cdot R \cdot B(\theta,x)^{-1}$$

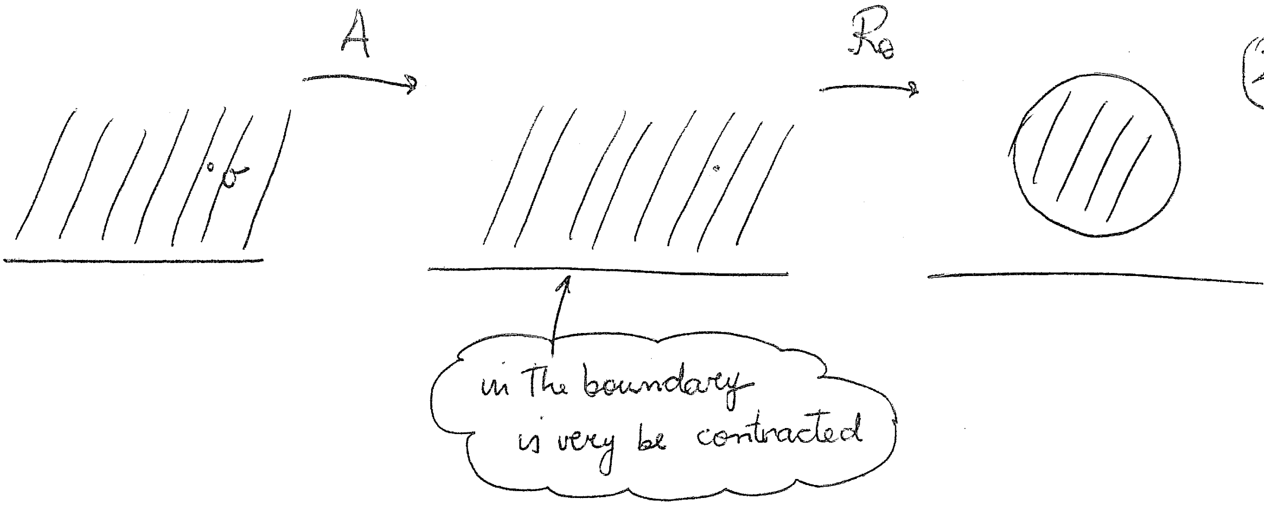
$$\Downarrow$$

A^n is bounded in L^1

$$\int \|A^n\| \, d\mu \leq C < +\infty$$

<u>Proof</u> (Idea): $\mathrm{Im}(\theta) = \varepsilon$

$$\lambda(\theta) = -\frac{1}{2}\int \ln(\text{contraction at } u)\, du \quad \text{(Poincaré Metric)}$$



$$\lambda(\theta) \geq \varepsilon \cdot \int \frac{1+|u|^2}{2 \cdot \operatorname{Im} u} \, du$$

$$c \geq \frac{\lambda(\theta + i\varepsilon)}{\varepsilon} \geq \int \frac{1+|u|^2}{2 \operatorname{Im} u} \, du$$

$$\frac{d}{d\varepsilon} \lambda(\theta + i\varepsilon) \Big|_{\varepsilon=0} < +\infty$$

The Hilbert transform of λ is monotonic.

- λ is harmonic
- $\lambda + iN$ is holomorphic.

"integrated density of states"

measures amount of rotations

$R_\theta A(f^{n-1}(x)) \, R_\theta A(f^{n-2}(x)) \cdots R_\theta A(x)$ always goes in same direction, i.e., $\forall v \in \mathbb{P}\mathbb{R}^2 \ \frac{d}{d\theta} \arg\left((R_\theta A)^n(x) \right) \cdot v > 0$.

$$\lambda(\theta + i\varepsilon) = \lambda(\theta) + \varepsilon\,\partial$$

derivative perhaps is not def.

$$N(\theta + \varepsilon) = N(\theta) + \varepsilon\,\partial$$

There are the same terms because of Cauchy-Riemann

$\partial = \dfrac{\partial N}{\partial \theta}$ since N is monotonic

$\dfrac{\partial N}{\partial \theta} < \infty$ a.e.

Examples:

1) Dinaburg - Sinai

$f(x) = x + \alpha$, $\alpha \notin \mathbb{Q}$, α Diophantine

$A \subset C^{\omega}(\mathbb{R}/\mathbb{Z}, SL(2,\mathbb{R}))$

A is close to $A_* \in SL(2,\mathbb{R})$

Theorem: $\exists$ positive measure, set of θ such that $\lambda = 0$.

Use KAM Theorem

diophantine quasiperiodic motion is stable in parametrized families

$(\alpha, R_\theta A) : \mathbb{R}/\mathbb{Z} \times \mathbb{P}\mathbb{R}^2 \longrightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{P}\mathbb{R}^2$

$R_\theta A \sim R_\theta A_*$ elliptic

$(\alpha, R_\theta A_*)$ is a rigid rotation of 2-torus

PROBLEM: Zero Lyapunov exponents in positive measures of parameters

$\Downarrow$

almost periodicity (modulo triviality)

$f : X \longrightarrow X$ is homeomorph.

$A : X \longrightarrow SL(2,\mathbb{R})$ continuous

$\operatorname{supp}\mu = X$, $\lambda = 0$ for $C \cdot A$ for positive measure set of $C \in SL(2,\mathbb{R})$

$\Downarrow$

$(A(f^n(x)))_{n\in\mathbb{Z}}$ is almost periodic $\forall x$

$(A(f^n(x)))_{n\in\mathbb{Z}}$ is almost period.

$SO(2,\mathbb{R})$

Theorem: $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $A \in C^\omega(\mathbb{R}/\mathbb{Z}, SL(2,\mathbb{R}))$. For a.e. θ with $\lambda = 0$, $(f, R_\theta A)$ is analytically conjugate to $SO(2,\mathbb{R})$.

(A., Krikorian; A. Fayad, Krikorian)

Renormalization Argument to bring to local situation (either close to constant or to simple $x \longmapsto R_{nx}$).

Then by local analysis of KAM flavor if close to constant.

Complex analysis near $x \longmapsto R_{nx}$.

non standard KAM works with Liouville α

Trieste, 11/07/2008.

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$ $\quad f(x,y,z,w) = (x+\alpha, y+x, z+y, w+z)$

<u>Theorem</u>: Skew-shift f. $\{\lambda > 0\}$ is dense in $C^\infty(X, SL(2,\mathbb{R}))$

<u>Question about</u>:

$R_\theta A$ has an interval with $\lambda = 0$

$\Downarrow$

continuous conjugation to rotations

depending analytically on θ

$$B : I \times X \longrightarrow SL(2,\mathbb{R}) \quad \begin{cases} \text{continuous in } \theta, x \\ \text{analytic in } \theta \end{cases}$$

$$B(\theta, f(x))\, R_\theta A(x)\, B(\theta,x)^{-1} \in SO(2,\mathbb{R})$$

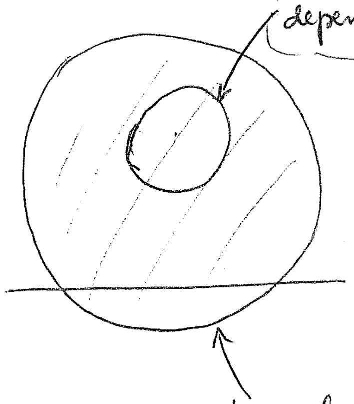
for skew-shifts B is C^∞.

The "polynomial-type" (subexponential growth of derivative).

Allows to go from analyticity on θ to C^∞ on x.

depend. smooth

θ



depend. continuous

- $\operatorname{Im}\theta > 0$ estimate dependence on x by U.H.
- together with KOTANI continuity
- Allows interpolation to control I.

PROBLEM: f a C^∞ diffeomorphism,

μ a fully supported ergodic measure

Then $\{\lambda>0\}$ is dense in $C^\infty(X, SL(2,\mathbb{R}))$?

$$A_E(x) = \begin{pmatrix} E - 2K\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix} \qquad X = \mathbb{R}/\mathbb{Z} \qquad f(x) = x+\alpha$$

(Almost Mathieu cocycles)

Theorem: $\forall \alpha \in \mathbb{R}\setminus\mathbb{Q}$, $K\neq 0$, $\{E, \lambda>0\} \cup \{E, A_E \text{ is U.H.}\}$ is dense in $\mathbb{R}$

$$\Updownarrow$$

Spectrum of almost Mathieu operator is a Cantor set

$x \mapsto x+\alpha$, $\alpha \in \mathbb{R}\setminus\mathbb{Q}$

What to do with $\{\lambda = 0\}$ in a positive measure set ?

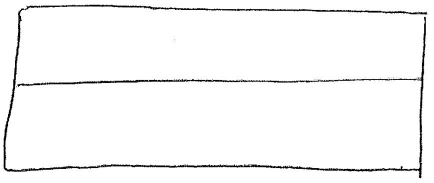
$\exists\ B \in L^2(X, SL(2,\mathbb{R}))$

Theorem: (A., Krikorian) $x \mapsto x+\alpha$, $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, $A\in C^\omega(\mathbb{R}/\mathbb{Z}, SL(2,\mathbb{R}))$

$\frac{p_k}{q_k}$ sequence of aproximats. $A_{q_k}(x)$ is getting close to a cocycle

$\longrightarrow$

of rotations with linear dependence on x (In scales like $1/q_k$, almost every where). More precisely, a.e. $x_0 \in \mathbb{R}/\mathbb{Z}$ $A_{q_k}(x_0 + x/q_k)$ is close to $B_{x_0}^{-1} A_{kx+y} B_{x_0}$ as a function of x (uniformly on compacts of $\mathbb{C}$).

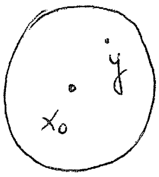


} h domain of analiticity of A

$\Downarrow$

$q_k h$ is the domain of

$A_{q_k}(x_0 + x/q_k)$



$$A_{q_k}(y) \leq \exp\,(q_k |y - x|)$$

(if so, can take limits $\tilde{A}$,

$$\|\tilde{A}(y)\| \leq c \cdot \exp |y| \;)$$

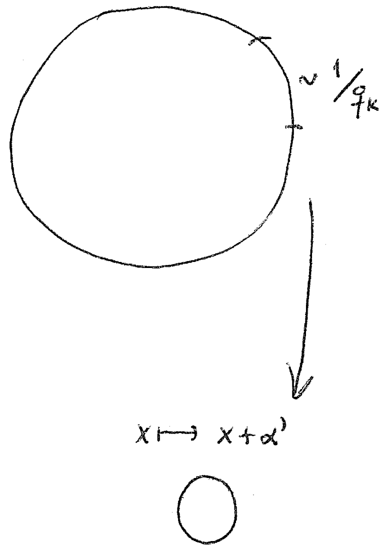
$$A(x_0 + (q_k - 1)\alpha) \cdot \ldots \cdot A(x_0 + \alpha) \cdot A(x_0)$$

comparing : $A(y + (q_k - 1)\alpha) \cdot \ldots \cdot A(y+\alpha)\, A(y)$

The difference involves $|y - x_0|$ (due to Lipschitz dependence of A on y) and matrix products (which can be estimated by L^2-conjugacy).

$$\sum_{n=0}^{N-1} \| B(x_0 + n\alpha) \|^2 \leq CN$$ controlled by maximal ergodic theorem.

With this Theorem in hand, reduces by renormalization



α' related to α by Gauss map

First return cocycle is close to linear.

GLUING PROCEDURE

New cocycle close to $A(x) = R_{kx+y}$ (for some $k \in \mathbb{Z}$)

1) Homotopic to constant (i.e., $k=0$)

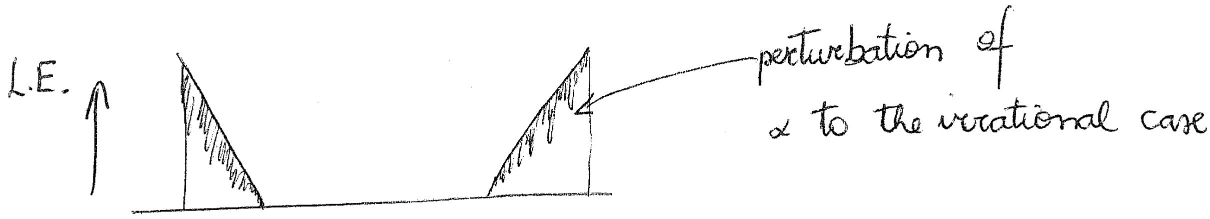
(Complicated case)

Theorem (Bourgain, Jitomirskaya): $x \mapsto x+\alpha$, $A \in C^\omega$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. $\lambda(R_\theta A)$ is continuous as a function of θ.

Important Remark: This is a delicate theorem because the continuity is not uniform when α changes (specially for α rational).

- Possible graph of Lyapunov exp. for α rational

$\longrightarrow$



Never expect better than ½ · Holder.

Example: constant cocycle $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ $t \neq 0$

Theorem (Bourgain, Jitomirskaya): C^ω near constants, $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Dichotomy: $\forall \theta$, either U.H. or $\lambda = 0$.

- α Diophantine fixed (earlier results by Eliasson).

For generic θ with $\lambda = 0$, $(f, R_\theta A)$ is not conjugate to rotations.

(2) Non-homotopic to constant (A., Krikorian)

$\mathcal{M} = \{ A$ analytic, A C^1 close to $R_{kx+y} \}$ $(k \neq 0)$

Remark: Remind that A., Javier showed during the course that $\nexists$ U.H. here.

⟶

Theorem: For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $A \mapsto \lambda(A)$ is analytic.
If α changes, $A \mapsto \lambda(A)$ is C^∞.

Theorem: $A \in \mathcal{M}$.

$$\lambda = 0 \iff (f, A) \text{ is analytically conjugated to } SO(2, \mathbb{R}).$$

Theorem: $A \in \mathcal{M}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, projective action is minimal

$$\mathbb{R}/\mathbb{Z} \times \mathbb{P}\mathbb{R}^2 \longrightarrow \mathbb{R}/\mathbb{Z} \times \mathbb{P}\mathbb{R}^2 .$$
$$(x, y) \longmapsto (x + \alpha, A(x) \cdot y)$$

(Idea: Schwarz reflection principle).