CONSTRUCTION OF STRONG STABLE MANIFOLDS FROM A NON-UNIFORM DOMINATED SPLITTING

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Abstract. We construct the strong stable manifold of a trajectory that has a non-uniform dominated splitting, rather than the non-uniformly hyperbolic splitting. As a consequence, we show that having a high-quality non-uniform dominated splitting leads to high quality strong stable manifolds, e.g. they are long and smooth.

1. Introduction

Suppose that $f_1, f_2, \ldots$ is a sequence of diffeomorphisms of a manifold M. Then we write $f^n = f_n \cdots f_1$ for their composition. In this note, we demonstrate that if $\{D_x f^n\}$ has a (C, κ, ϵ)-subtempered index ℓ dominated splitting, then the strong stable manifold at a point x exists and has a bound on its C^k norm in a uniformly large neighborhood of x. This notion is made precise by the following definition.

Definition 1.1. Suppose that M is a Riemannian manifold and that N is an immersed submanifold. We say that N is (C, k)-good at a point $x \in N$, if when we write N as the graph over its tangent space in an exponential chart in M, $\phi\colon U \subset T_xN \to (T_xN)^\perp$, $\phi(0) = 0$, then on a ball of radius C^{-1} in T_xN, $B(0, C^{-1}) \subset T_xN$, $\|\phi|_{B(0,C^{-1})}\|_{C^k} \le C$.

We also make precise the type of splitting that we consider here. For a sequence of matrices (A_n) we write $A^{i,j}$ for the product $A_{j+i-1} \cdots A_i$.

Definition 1.2. Suppose that $(E_n, \|\cdot\|_n)_{n\ge1}$ is a sequence of inner product spaces, and that $(A_n)_{n\ge1}$, $A_n\colon E_n \to E_{n+1}$ is a sequence of linear maps. We say that this sequence has (C, κ, ϵ)-*subtempered dominated splitting* of index ℓ if there exists a sequence of decompositions $E_i = V_i \oplus W_i$, $A_k(V_k) = V_{k+1}$, $A_k(W_k) = W_{k+1}$ with $\dim(V_i) = d - \ell$ such that for all non-zero $v \in V_k$ and $w \in W_k$,

$$\|A^{k,j}v\|/\|v\| \le e^C e^{-\kappa j} e^{k\epsilon} \|A^{k,j}w\|/\|w\|, \tag{1.1}$$

for all unit $v \in V_k$,

$$\|A^{k,j}v\|/\|v\| \le e^C e^{-\kappa j} e^{\epsilon k}, \tag{1.2}$$

and

$$\angle(A^k V_1, A^k W_1) \ge e^{-C} e^{-\epsilon k}. \tag{1.3}$$

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Our main result is that this type of splitting leads to quantitative control on the strong stable manifolds. When we say *strong stable manifold* in the following theorem, we mean only what the theorem is providing: a disk with the given contraction properties.

Theorem 1.3. Fix a Riemannian manifold M, $C > 0$, $\kappa > 0$, $k \geq 2$, and $\Lambda > 0$. Then there exist $\epsilon_0 > 0$ and $m > 0$ such that for any $0 < \epsilon < \epsilon_0$, there exists $\hat{C}_1, \hat{C}_2, \hat{C}_3 > 0$ such that if f^n is a sequence of maps with C^k norm bounded by e^{Λ}, $x \in M$, and $D_x f^n$ has a (C, κ, ϵ)-subtempered dominated splitting of index ℓ, then $(d-\ell)$-dimensional strong stable manifold $W^{ss}(x)$ is defined at this point, is C^{k-1}, is $(\hat{C}_1, k-1)$-good in the sense of Definition 1.1, and is tangent at x to the contracting subspace of the dominated splitting. This means that:

(1) There is an embedded C^{k-1}-disk D of dimension $d-\ell$ that is $(\hat{C}_1, k-1)$-good at x.
(2) $T_x D = V_1$, where V_1 is the contracting direction of the subtempered dominated splitting.
(3) Furthermore, any two points $y, z \in D$ satisfy that

$$d(f^n x, f^n y) = o(\|D_x f^n | V_1\| e^{m\epsilon}). \tag{1.4}$$

(4) Within a ball of radius $\hat{C}_2 > 0$ of x, $B_{\hat{C}_2}(x)$, the only points z satisfying

$$d(f^n x, f^n y) \leq \hat{C}_3 \|D_x f^n | V_1\| e^{m\epsilon},$$

are the points in $D \cap B_{\hat{C}_2}(x)$.

Similar results are often proven in Pesin theory. However, they usually impose a stronger requirement. The definition below is taken from [BP07, Def. 1.2.1]. See also [LQ95, Sec. III.1].

Definition 1.4. We say that f has a $(\lambda, \mu, \epsilon, C)$-splitting at a point x if there is a decomposition $T_x M = E_1^{\leq \lambda} \oplus E_1^{\geq \mu}$ such that if we set $E^*_{n+1} = Df^n E^*$, then there is a constant C such that:

(1) For $v \in E_n^{\leq \lambda}$ and $m \in \mathbb{N}$,

$$\|Df^{n,m} v\| \leq e^{C+\epsilon n - \lambda m} \|v\|.$$

(2) For $u \in E_n^{\geq \mu}$,

$$\|Df^{n,m} v\| \geq e^{-C+\epsilon n + \mu m} \|u\|.$$

(3) The angle between these subspaces is not decaying too fast:

$$\left| \angle (E_n^{\leq \lambda}, E_n^{\geq \mu}) \right| > e^{-C} e^{-\epsilon n}.$$

Many references prove Theorem 1.3 under the assumption of existence of $(\lambda, \mu, \epsilon, C)$-splitting (with the bounds on smoothness depending on the constant C from the Definion 1.4). See, for example [BP07, Sec. 8.4], [LQ95, Thm. III.3.2], [BV12], [BRV20]. There is also a small number of references that work with dominated splittings. For example, see [Cro+24]. However, these constructions either use a uniform hypothesis on the domination or are restricted to a particular dimension.

It turns out that in the setting of random dynamics it is often easier to check the conditions of Definition 1.2 that of Definition 1.4, see e.g. [DDZ26; Zha19]. Thus it is convenient to have a result which provides control on the strong stable manifolds in the setting of Definition 1.2. Even then both definitions apply, it might turn out that a trajectory has a bad constant C in Definition 1.4, but has a good constant C in Definition 1.2. In this case we are able to conclude that a better stable manifold exists than is promised by the classical theory: it may be larger and smoother.

Due to the characterization of strong stable manifolds in parts (3) and (4) of our main theorem we have the following immediate corollary.

Corollary 1.5. If a trajectory has $(\lambda, \mu, \epsilon, C)$-splitting in the sense of Definition 1.4, then the Pesin strong stable manifolds constructed in the above references coincides with the strong stable manifolds given by Theorem 1.3.

We give below a relatively detailed proof of Theorem 1.3. In some of the parts that are most standard, we do not give the full details. For the parts that are most different from the usual argument, i.e. Steps 1 and 4, we give full details.

The argument has 5 steps:

(1) Construct an adapted metric on $T_{f^n(x)}M$, similar to the usual Lyapunov metric used in Pesin theory. This is in Section 2.
(2) Use the adapted metric to construct Lyapunov charts ψ_n on a small neighborhood of $f^n(x)$. This is in Section 3.
(3) Extend the dynamics as seen in the Lyapunov charts to "globalized" dynamics $\mathbb{R}^d \to \mathbb{R}^d$ that are perturbations of linear maps. This is in Section 4.
(4) Obtain quantitative C^k-estimates on the graph transform for diffeomorphisms close to a linear map $\mathbb{R}^d \to \mathbb{R}^d$. (Section 5).
(5) Using the Lyapunov charts and the associated globalized dynamics, deduce from the previous step that stable manifolds exist in these charts, and hence that the strong stable manifolds also exist in M. (Section 6).

Remark 1.6. Throughout the proof we will write d_{ss} and d_c for the dimensions $d - \ell$ and ℓ respectively. Later in the proof, we when are doing the graph transform, we will consider the inverse of this dynamics, in which as we will write d_{uu} and d_c for these numbers.

The last section of the paper, Section 7 contains an application of our result to random dynamics. Under appropriate assumptions on the (C, κ, ϵ)-good trajectories we prove a result concerning the regularity of the disintegration of the volume along the strong stable lamination restricted to this set.

2. Step 1. Construction of the adapted metric

There appear to be few places in the literature where an adapted metric is constructed for dynamics with a dominated splitting. One of the papers addressing this directly is [Gou07], which constructs an adapted metric for systems with a uniform dominated splitting (i.e. $\epsilon = 0$ in Definition 1.2) of a cocycle over a homeomorphism of a compact metric space. See also [Cro+24] which carries out a construction of invariant manifolds in the presence of a dominated splitting for diffeomorphisms of a surfaces. The paper [Gou07] constructs a function that non-uniformly "separates" the norms on the two bundles, i.e. this function is larger than norm on one bundle and smaller than the conorm on the other. Gourmelon then constructs an adapted metric using this separating function. In our argument, we divide the sequence of matrices into blocks, where a block is long enough to witness the domination. Then we define the separating function for the matrices in that block by letting the separating function equal the average of the total domination over the block. Our strategy is similar, but the proof is more complicated because our assumptions on the dominated splitting are not uniform along the entire trajectory.

Before proceeding to the proof, we will give a lemma that shows how the norms change when one constructs an adapted metric, either going forwards or backwards in time.

Lemma 2.1. Suppose that A_n is a sequence of linear maps $\mathbb{R}^d \to \mathbb{R}^d$, and η_i is a sequence of numbers. Let $\eta(i,j) = \prod_{\ell=i}^{j-1} \eta_\ell$. Consider a forward-looking metric

$$(\|\xi\|_i^s)^2 = \sum_{\ell=0}^{\infty} \eta(i, i+\ell)^2 \|A^{i,\ell}\xi\|^2. \tag{2.1}$$

as well as the backwards-looking metric

$$(\|\xi\|_i^u)^2 = \sum_{\ell=0}^{i} \eta(i-\ell, i)^2 \|[A^{i-\ell,\ell}]^{-1}\xi\|^2. \tag{2.2}$$

Then

$$\|A_i\xi\|_{i+1}^s \leq \eta_i^{-1}\|\xi\|_i^s \quad \text{and} \quad \|A_i\xi\|_{i+1}^u \geq \eta_i\|\xi\|_i^u.$$

Proof. The proofs are routine, but we give them here for completeness.

$$\begin{aligned}
(\|A_i\xi\|_{i+1}^s)^2 &= \sum_{\ell=0}^{\infty} \eta(i+1, i+1+\ell)^2 \|A^{i+1,\ell}A_i\xi\|^2 = \sum_{\ell=0}^{\infty} \eta(i+1, i+1+\ell)^2 \|A^{i,\ell+1}\xi\|^2 \\
&= \sum_{\ell=0}^{\infty} \frac{\eta(i+1, i+1+\ell)^2}{\eta(i, i+\ell+1)^2} \eta(i, i+\ell+1)^2 \|A^{i,\ell+1}\xi\|^2 \\
&= \eta_i^{-2} \sum_{\ell=0}^{\infty} \eta(i, i+\ell+1)^2 \|A^{i,\ell+1}\xi\|^2 = \eta_i^{-2}\left((\|\xi\|_i^s)^2 - \|\xi\|^2\right) \leq \eta_i^{-2}(\|\xi\|_i^s)^2.
\end{aligned}$$

For the other norm:

$$
\begin{aligned}
(\|A_i\xi\|^u_{i+1})^2 &= \sum_{\ell=0}^{i+1} \eta(i+1-\ell,\ell)^2 \|[A^{i+1-\ell,\ell}]^{-1} A_i\xi\|^2 \\
&= \|A_i\xi\|^2 + \sum_{\ell=1}^{i+1} \eta(i-(\ell-1),\ell)^2 \|[A^{i-(\ell-1),\ell-1}]^{-1}\xi\|^2 \\
&\geq \sum_{\ell=1}^{i+1} \frac{\eta(i-(\ell-1),\ell)^2}{\eta(i-(\ell-1),\ell-1)^2} \eta(i-(\ell-1),\ell-1)^2 \|[A^{i-(\ell-1),\ell-1}]^{-1}\xi\|^2 \\
&= \eta_i^2 \sum_{\ell=1}^{i+1} \eta(i-(\ell-1),\ell-1)^2 \|[A^{i-(\ell-1),\ell-1}]^{-1}\xi\|^2 = (\eta_i\|\xi\|^u_i)^2.
\end{aligned}
$$

This concludes the proof. □

Remark 2.2.

Note that in the course of the proof of Lemma 2.1, if we carry through the dropped term to the end of the proof, then we obtain the following exact expressions:

$$(\|A_i\xi\|^u_{i+1})^2 = \|A_i\xi\|^2 + \eta_i^2(\|\xi\|^u_i)^2 \tag{2.3}$$

$$(\|A_i\xi\|^s_{i+1})^2 = \eta_i^{-2}\left((\|\xi\|^s_i)^2 - \|\xi\|^2\right) \tag{2.4}$$

From these we can obtain bounds on the norm and conorm of A_i in the new metrics. First, for the backwards looking metric, $\|\cdot\|^u_i$, by definition of the metric, $\|\xi\|_i \leq \|\xi\|^u_i$. Hence,

$$\eta_i^2(\|\xi\|^u_i)^2 \leq (\|A_i\xi\|^u_{i+1})^2 \leq \|A_i\|^2(\|\xi\|^u_i)^2 + \eta_i^2(\|\xi\|^u_i)^2$$

Then for the forwards looking metric, we first make an auxiliary observation:

$$\frac{1}{[1+\eta_i^2 m(A_i)^2]}(\|\xi\|^s_i)^2 \geq \|\xi\|^2_i. \tag{2.5}$$

This follows by extracting the first two terms:

$$(\|\xi\|^s_i)^2 \geq \|\xi\|^2_i + \eta_i^2\|A_i\xi\|^2_{i+1} \geq (1+\eta_i^2 m(A_i)^2)\|\xi\|^2_i.$$

Plugging (2.5) into (2.4) we find that

$$
\begin{aligned}
(\|A_i\xi\|^s_{i+1})^2 &\geq \eta_i^{-2}\left([1-[1+\eta_i^2 m(A_i)^2]^{-1}])(\|\xi\|^s_i)^2 + \frac{1}{[1+\eta_i^2 m(A_i)^2]}(\|\xi\|^s_i)^2 - \|\xi\|^2_i\right) \\
&\geq \eta_i^{-2}[1-[1+\eta_i^2 m(A_i)^2]^{-1}])(\|\xi\|^s_i)^2.
\end{aligned}
$$

Thus as the other bounds were already contained in Lemma 2.1 we obtain,

$$
\begin{aligned}
\eta_i^{-1}\sqrt{1-[1+\eta_i^2 m(A_i)^2]^{-1}}\|\xi\|^s_i \leq \|A_i\xi\|^s_{i+1} \leq \eta_i^{-1}\|\xi\|^s_i \\
\eta_i\|\xi\|^u_i \leq \|A_i\xi\|^u_{i+1} \leq \sqrt{\|A_i\|^2+\eta_i^2}\|\xi\|^u_i.
\end{aligned}
\tag{2.6}
$$

These will be useful later, because as long as the η_i and A_i are uniformly bounded above and below, then A_i in the adapted metric will also be uniformly bounded above and below.

We now turn to the construction of the adapted metric.

Lemma 2.3. (Construction of the adapted metric) Fix $d, \ell, \Lambda, \kappa > 0$. Then there exists a constant $c_0 = C_{\Lambda,\kappa,d,\ell}$ such that the following holds. Suppose that

$$0 < \epsilon < \kappa/6 \tag{2.7}$$

$$4\epsilon\kappa^{-1}\Lambda + \epsilon - \kappa/4 < -\kappa/5, \tag{2.8}$$

$(E_n, \|\cdot\|_n)$ is a sequence of d-dimensional inner product spaces, and A_n is a sequence of linear maps $A_n\colon E_n \to E_{n+1}$ with a (C, κ, ϵ)-subtempered dominated splitting of index ℓ with $V_i \oplus W_i = E_i$ and $\max\{\|A_i\|, \|A_i^{-1}\|\} \le e^{\Lambda}$. Then there exist inner products $\|\cdot\|_i'$ on E_i with the following properties:

(1) For any vector $\xi_V \in V_i$, $\|A_i \xi_V\|_{i+1}' \le e^{-3\kappa/20}\|\xi_V\|_i'$.
(2) For any unit vectors $\xi_V \in V_i$ and $\xi_W \in W_i$,

$$\|A_i\xi_W\|_{i+1}' \ge e^{\kappa/4}\|A_i\xi_V\|_{i+1}'. \tag{2.9}$$

(3) The adapted metrics are comparable to the original metric. Namely,

$$\frac{1}{\sqrt{2}}\|\xi\|_i \le \|\xi\|_i' \le e^{c_0 C + c_0\epsilon i + c_0}\|\xi\|_i. \tag{2.10}$$

Proof. We will define the metric $\|\cdot\|_i'$ on V_i and W_i individually, and then extending to $V_i \oplus W_i$ by declaring the two to be orthogonal. We proceed in two steps. First, we will construct an auxiliary metric on V_i that makes the contraction uniform. Then using this metric we will identify certain blocks that we wish to average the domination over, which leads us to the definition of the functions η_i^V, η_i^W in (2.16). After that, some further bookkeeping verifies the claims in the lemma.

First, we define a metric $\|\cdot\|_i''$ on V_i similarly to the usual definition of a Lyapunov metric so that the contraction on V_i is uniform with respect to $\|\cdot\|_i''$. For $\xi_V \in V_i$, define

$$(\|\xi_V\|_i'')^2 = \sum_{\ell=0}^{\infty} e^{(4\kappa/5)\ell}\|A^{i,\ell}\xi_V\|_{i+\ell}^2. \tag{2.11}$$

Note that with respect to this metric for $\xi_V \in V_i$,

$$\begin{aligned}(\|A_i\xi_V\|_{i+1}'')^2 &= \sum_{\ell=0}^{\infty} e^{(4\kappa/5)\ell}\|A^{i+1,\ell}A_i\xi_V\|_{i+1+\ell}^2 \\ &= \sum_{\ell=1}^{\infty} e^{(4\kappa/5)\ell}e^{-4\kappa/5}\|A^{i,\ell}\xi_V\|_{i+\ell}^2 \le e^{-4\kappa/5}(\|\xi_V\|_i'')^2.\end{aligned} \tag{2.12}$$

When we construct the metric $\|\cdot\|'$ below, we will need to compare the norm of $A^{i,\ell}$ in $\|\cdot\|$ and $\|\cdot\|''$. Note that ℓ appears only in the right hand bound in (2.13).

Lemma 2.4. There exists some c_κ such that for the $\|\cdot\|_i''$ metric just constructed

$$e^{-\epsilon i - C - c_\kappa}\|A^{i,\ell}|_{V_i}\|_{i+\ell} \le \|A^{i,\ell}|_{V_i}\|''_{i+\ell} \le \|A^{i,\ell}|_{V_i}\|_{i+\ell} e^{\epsilon(i+\ell)+C+c_\kappa}. \tag{2.13}$$

Proof. To estimate the terms in the product, we compare the norms $\|\cdot\|_i''$ and $\|\cdot\|$ on V_i. For this, first note that by extracting the first term in (2.11) we get $\|\xi_V\|_i'' \ge \|\xi_V\|_i$. For the opposite inequality, note that for $\xi_V \in V_i$, by (1.2),

$$(\|\xi_V\|_i'')^2 \le \sum_{\ell=0}^{\infty} e^{(4\kappa/5)\ell} e^{-2\kappa\ell+2\epsilon i+2C}\|\xi_V\|_i^2 \le C_\kappa^2 e^{2\epsilon i+2C}\|\xi_V\|_i^2. \tag{2.14}$$

These inequalities allow us to bound $\|A^{i,j-i}|_{V_i}\|_i''$ in terms of $\|A^{i,j-i}|_{V_i}\|_i$, namely

$$\begin{aligned}\|A^{i,\ell}|_{V_i}\|'' = \max_{\xi_V\neq 0}\frac{\|A^{i,\ell}\xi_V\|''_{i+\ell}}{\|\xi_V\|_i''} &= \max_{\xi_V\in V_i\setminus\{0\}}\frac{\|A^{i,\ell}\xi_V\|''_{i+\ell}}{\|A^{i,\ell}\xi_V\|_{i+\ell}}\frac{\|A^{i,\ell}\xi_V\|}{\|\xi_V\|}\frac{\|\xi_V\|_i}{\|\xi_V\|_i''}\\ &\le C_\kappa e^{\epsilon(i+\ell)+C}\|A^{i,\ell}|_{V_i}\|.\end{aligned}$$

Similarly,

$$\|A^{i,\ell}|_{V_i}\|'' = \max_{\xi_V\in V_i\setminus\{0\}}\frac{\|A^{i,\ell}\xi_V\|''_{i+\ell}}{\|A^{i,\ell}\xi_V\|_{i+\ell}}\frac{\|A^{i,\ell}\xi_V\|}{\|\xi_V\|}\frac{\|\xi_V\|_i}{\|\xi_V\|_i''} \ge 1\cdot\|A^{i,\ell}\|\cdot C_\kappa^{-1}e^{-\epsilon i - C}.$$

This finishes the proof of (2.13). □

We now define the functions η_i that we use to construct the final adapted metric. For this we divide the sequence A_i into blocks corresponding to a certain domination time for the splitting. Let κ be as before and fix $\kappa' = \kappa/3$.

Given an index i_k, let t_k be the first $t\in\mathbb{N}$ such that

$$\|A^{i_k,t}|_{V_{i_k}}\|'' \le e^{-\kappa' t} m(A^{i_k,t}|_{W_{i_k}}) \tag{2.15}$$

where this co-norm on the right is with respect to the original metric.

We then partition the sequence of matrices A_i into consecutive blocks indexed by k. The kth block will begin at index i_k and contains t_k matrices. Hence $i_k + t_k$ is equal to i_{k+1}. Then for $i_k \le i < t_k$, we define the functions:

$$\eta_i^V = \left[(\|A^{i_k,t_k}|_{V_{i_k}}\|'')^{1/t_k} e^{\kappa/4}\right]^{-1} \tag{2.16}$$

$$\eta_i^W = (\|A^{i_k,t_k}|_{V_{i_k}}\|'')^{1/t_k} e^{\kappa/2}. \tag{2.17}$$

Observe that, in view of (2.12)

$$(\eta_i^V)^{-1} \le e^{-\frac{2}{5}\kappa} e^{\kappa/4} = e^{-3\kappa/20}. \tag{2.18}$$

Next, we estimate the size of j_k. In order for the event in (2.15) to occur, in view of (2.13), it suffices to know that

$$e^{i_k\epsilon - \kappa t + C} \le e^{-\kappa' t} e^{-\epsilon(i_k+t)-C-c_\kappa}.$$

Thus (2.15) occurs provided that

$$t \geq \frac{2C + 2\epsilon i_k + c_\kappa}{\kappa - \kappa' - \epsilon}.$$

In particular,

$$t_k \leq \left\lceil \frac{2C + 2\epsilon i_k + c_\kappa}{\kappa - \kappa' - \epsilon} \right\rceil.$$

Hence as we choose $\kappa' = \kappa/3$ and $\epsilon < \kappa/6$, it follows that:

$$t_k \leq \frac{4C}{\kappa} + \frac{4i_k\epsilon}{\kappa} + c'_\kappa.$$

In particular, this implies that if i is some index and i_k is the beginning of the next block, then:

$$i_k \leq i + \frac{4C}{\kappa} + \frac{4\epsilon i}{\kappa} + c'_\kappa. \tag{2.19}$$

We now define the metric $\|\cdot\|'_i$ that is provided by the lemma. On V_i we define $\|\cdot\|'_i$ by applying Lemma 2.1 to the original metric $\|\cdot\|_i$ using the function η_i^V and the forwards metric construction (2.1). We define the metric $\|\cdot\|'_i$ on W_i using η_i^W and the backwards metric construction (2.2). Finally, We declare V_i and W_i to be orthogonal.

We now verify that the conclusions of the lemma hold for $\|\cdot\|'_i$.

Proof of (1). From Lemma 2.1 and (2.18), it follows that for a unit vector $\xi \in V_i$,

$$\|A_i\xi\|'_{i+1} \leq (\eta_i^V)^{-1}\|\xi\|'_i = e^{-\frac{3}{20}\kappa}\|\xi\|'_i.$$

Proof of (2). Suppose that $\xi^V \in V_i$ and $\xi^W \in W_i$ are $\|\cdot\|'_i$ unit vectors, then from Lemma 2.1, it follows that:

$$\|A_i\xi^V\|'_{i+1} \leq (\|A^{i_k,t_k}\|'')^{1/t_k}e^{\kappa/4} \quad \text{and} \quad \|A_i\xi^W\|'_{i+1} \geq (\|A^{i_k,t_k}\|'')^{1/t_k}e^{\kappa/2},$$

which gives the desired bound.

Proof of (3). This part is the most complicated. First, we will compare the metrics $\|\cdot\|_i$ and $\|\cdot\|'_i$ restricted to the V_i and W_i. Due to the construction in Lemma 2.1, the nontrivial estimates are those where we bound $\|\cdot\|'_i$ above in terms of $\|\cdot\|_i$.

We begin with the V_i. First, for $\xi \in V_i$, the estimate $\|\xi\|_i \leq \|\xi\|'_i$ is immediate as this is the first term in the sum defining $\|\cdot\|'_i$. The upper bound is more complicated and relies on Lemma 2.4.

Claim 2.5. There exists $c_1 = C_{\Lambda,\kappa}$ such that for for fixed κ, Λ as above for any $\xi \in V_i$,

$$\|\xi\|'_i \leq e^{c_1\epsilon i + c_1 C + c_1}\|\xi\|_i. \tag{2.20}$$

Proof. By definition, $(\|\xi\|'_i)^2 = \sum_{j\geq 0}\eta^V(i,i+j)^2\|A^{i,j}\xi\|^2_{i+j}$. So, it will suffice to obtain an estimate on $\eta^V(i,i+j)\|A^{i,j}|_{V_i}\xi\|$. We will obtain an estimate of the form: $\eta(i,i+j)\|A^{i,j}\xi\|_{i+j} \leq e^{C_{\Lambda,\kappa}+C_{\Lambda,\kappa}C - j\kappa/4}\|\xi\|_i$.

Take the product $A^{i,j} = A_{i+j-1} \cdots A_i$ and partition it according to the blocks the matrices lie in. In other words, if $i_k \le i + j' < i_{k+1}$ then the matrix $A_{i+j'}$ lies in the kth block. $A^{i,j}$ may not have all the matrices of each block that it meets. Thus we partition $A^{i,j}$ into some number $q+1$ of full blocks, where we have all the matrices between indices i_k and i_{k+1}, as well as (possibly) two partial blocks at the two ends. We denote the product of the matrices in the end blocks by B_L and B_R. Let $|L|$ and $|R|$ denote the lengths of these partial blocks. Hence

$$A^{i,j} = B_L A^{i_{k+q},t_{k+q}} A^{i_{k+q-1},t_{k+q-1}} \cdots A^{i_k,t_k} B_R.$$

Similarly, write η_L^V and η_R^V for the product of the η_i^V corresponding to the partial end blocks.

Thus if $\hat{\xi}_{V_{i_{k+n}}}$ is a unit vector in the direction of $A^{i,i_{k+n}-i}\xi$, then

$$\eta^V(i,i+j)\|A^{i,j}\xi\| \le \|B_L\|\eta_L^V \left[\prod_{n=0}^{q} \frac{\|A^{i_{k+n},t_{k+n}}\hat{\xi}_{V_{i_{k+n}}}\|}{\|A^{i_{k+n},t_{k+n}}|_{V_{i_{k+n}}}\|'' e^{t_{k+n}\kappa/4}}\right] \|B_R\|\eta_R^V$$

$$\le [\|B_L\|\eta_L^V] \left[\frac{\|A^{i_k,t}\hat{\xi}_{V_{i_k}}\|}{\|A^{i_k,t}|_{V_{i_k}}\|'' e^{t_k\kappa/4}}\right] [\|B_R\|\eta_R^V] \tag{2.21}$$

where $t = \sum_{n=0}^{q} t_{k+q}$. We now estimate the leftmost and rightmost terms. We can estimate $\|B_L\|$ and $\|B_R\|$ by the trivial bound e^Λ on the norm of the matrices. Due to the length of the end blocks:

$$\|B_L\| \le e^{t_{k+q+1}\Lambda}, \quad \|B_R\| \le e^{t_{k-1}\Lambda}.$$

Applying (2.13) to the definition of η_i^V, (2.16), we obtain an upper bound:

$$\eta_L^V \le \prod_{j=0}^{|L|} (e^{i_{k+q+1}\epsilon + C + c_\kappa} \|A^{i_{k+q+1},t_{k+q+1}}\|^{-1})^{\frac{1}{t_k}} e^{-\kappa/4} \le e^{\epsilon i_{k+q+1}+C+c_\kappa} e^{\Lambda|L|}.$$

Similarly,

$$\eta_R^V \le e^{\epsilon i + C + c_\kappa} e^{\Lambda|R|}. \tag{2.22}$$

Thus applying (2.19) to estimate the block length, (2.13) to the middle term of (2.21), and using the above estimates for the end blocks:

$$\eta^V(i,i+j)\|A^{i,j}\xi\| \le$$

$$e^{\left[\frac{8C}{\kappa}+\frac{8\epsilon(i_k+t)}{\kappa}+c'_\kappa\right]\Lambda} e^{\epsilon(i_k+t)+C+c_\kappa} \times \left[e^{\epsilon i_k + C + c_\kappa} e^{-t\kappa/4}\right] \times e^{\left[\frac{8C}{\kappa}+\frac{8\epsilon i}{\kappa}+c'_\kappa\right]\Lambda} e^{\epsilon i + C + c_\kappa}.$$

This simplifies to:

$$\eta^V(i,i+j)\|A^{i,j}\xi\| \le e^{[(16\Lambda\kappa^{-1}+3)C]} e^{\epsilon[8\kappa^{-1}\Lambda i + \epsilon i + 2i_k + 8\kappa^{-1}\Lambda i_k]} e^{[8\epsilon\kappa^{-1}\Lambda + \epsilon - \kappa/4](i_{k+q+1}-i_k)} e^{c''_{\kappa,\Lambda}}.$$

Applying again the bound on (2.19) on $i_k - i$, this implies that there are constants $c'''_{\Lambda,\kappa}$ such that:

$$\eta^V(i,i+j)\|A^{i,j}\xi\| \le e^{c'''_{\Lambda,\kappa}C + c'''_{\Lambda,\kappa}C + c'''_{\Lambda,\kappa}\epsilon i} e^{[4\epsilon\kappa^{-1}\Lambda + \epsilon - \kappa/4](i_{k+q+1}-i)}. \tag{2.23}$$

To conclude, we just need to estimate $i_{k+q+1} - i$ in the exponent. We consider two cases depending on how large j is relative to i.

(1) If $j < 2\epsilon/\kappa i$, then it follows that $1 \leq e^{2\epsilon i - \kappa j}$. So, in (2.23)

$$e^{[4\epsilon\kappa^{-1}\Lambda+\epsilon-\kappa/4](i_{k+q+1}-i)} \leq 1 \leq e^{2\epsilon i}e^{-\kappa j}.$$

(2) If $j \geq 2\epsilon\kappa^{-1}i$, then by (2.19), it follows that

$$(1+4\epsilon\kappa^{-1})i_{q+k+1} + c_4 \geq i_{k+q+2} \geq j+i.$$

Thus

$$i_{k+q+1} - i \geq j - [4\epsilon\kappa^{-1}i_{k+q+1}] - c_4.$$

Then as $i_{k+q+1} \leq j + i \leq (1+2\epsilon\kappa^{-1})i$, it follows that:

$$e^{[4\epsilon\kappa^{-1}\Lambda+\epsilon-\kappa/4](i_{k+q+1}-i)} \leq e^{[4\epsilon\kappa^{-1}\Lambda+\epsilon-\kappa/4]j}e^{-[4\epsilon\kappa^{-1}\Lambda+\epsilon-\kappa/4](1+2\epsilon\kappa^{-1})4\epsilon\kappa^{-1}i}e^{c^{(4)}_{\kappa,\Lambda}}.$$

Note that in the above expression the coefficient of i is a multiple of ϵ depending only on Λ and κ, and that the coefficient of j is negative and less than $\kappa/5$ by assumption (2.8).

Thus combining the two cases, we find that for a unit vector ξ, and any i and j, that there is a constant $C_{\kappa,\Lambda}$ such that:

$$\eta^V(i,i+j)\|A^{i,j}\xi\|_{i+j} \leq e^{c_{\Lambda,\kappa}+c_{\Lambda,\kappa}\epsilon+c_{\Lambda,\kappa}C-j\kappa/5}\|\xi\|_i.$$

Summing the series defining $\|\cdot\|_i'$ on V_i, we obtain (2.20), completing the proof of the claim. $\square$

Next, we will prove the analogous claim for the metric on the W_i bundle.

Claim 2.6. There exists $c_{\Lambda,\kappa}$ such that for fixed κ, η, Λ as above for any $\xi \in W_i$,

$$\|\xi\|_i' \leq e^{c_{\Lambda,\kappa}\epsilon i + c_{\Lambda,\kappa}C + c_{\Lambda,\kappa}}\|\xi\|_i. \tag{2.24}$$

Proof. The proof is analogous to the proof in the previous case, but we provide details as the definition of the metric is now different:

$$(\|\xi\|_i')^2 = \sum_{j\geq 0}\eta^W(i-j,i)^2\|[A^{i-j,j}]^{-1}\xi\|^2.$$

First we will obtain an estimate on $\eta^W(i-j,i)\|[A^{i-j,j}]^{-1}\xi\|$. For this we will similarly partition the matrices $A_i \cdots A_{i-j}$ into the blocks we used when defining the functions η_i^W in (2.15). (Things are simpler now because every block begins at an index less than i.) Then

$$\begin{aligned}
&\eta^W(i-j,i)\|[A^{i-j,j}]^{-1}\xi\| \\
&\leq \|B_L\|\eta^W(i-j,i_k)\left[\frac{\prod_{n=0}^q \|A^{i_{k+n},t_{k+n}}|_{V_{i_{k+n}}}\|''e^{\kappa/2}}{m(A^{i_k,i_{k+q}-i_k}|_{W_{i_k}})}\right]\|B_R\|\eta^W(i_{k+q+1},i) \\
&\leq \|B_L\|\eta^W(i-j,i_k)\left[\frac{\prod_{n=0}^q \|A^{i_{k+n},t_{k+n}}|_{V_{i_{k+n}}}\|''e^{\kappa/2}}{\prod_{n=0}^q m(A^{i_{k+n},t_{k+n}}|_{W_{i_k}})}\right]\|B_R\|\eta^W(i_{k+q+1},i).
\end{aligned}$$

Note that here we have used that $m(AB) \geq m(A)m(B)$, where m denotes the conorm of a linear transformation. We then estimate each of these three groups of terms.

First, we record an estimate on an individual η_i^W term. From definition (2.17), $\eta_i^W = (\|A^{i_k,t_k}|_{V_{i_k}}\|'')^{1/t_k}e^{\kappa/2}$. By (2.12), we know that the norm of A_i restricted to the V_i subspaces is at most $e^{-2\kappa/5}$ by (2.12). Thus it follows that $\eta_i^W \leq e^{-2\kappa/5}e^{\kappa/2} = e^{\kappa/10}$.

Due to the estimate on the block size (2.19), the total contribution from the left and right blocks to the product is of order $e^{\epsilon i c_{\kappa,\Lambda}+c_{\kappa,\Lambda}C+c_{\kappa,\Lambda}}$. We then consider the middle term. For this term, the blocks were chosen specifically in (2.15) so that:

$$\left[\frac{\prod_{n=0}^q \|A^{i_{k+n},t_{k+n}}|_{V_i}\|'' e^{\kappa/2}}{\prod_{n=0}^q m(A^{i_{k+n},t_{k+n}}|_{W_{i_k}})}\right] \leq e^{-(\kappa/3)(i_{k+q+1}-i_k)}.$$

Due to the estimate (2.19) on the length of the blocks, it follows that

$$(i_{k+q+1} - i) \geq j - 8\epsilon\kappa^{-1}i + 8\kappa^{-1}C + 2c'_\kappa.$$

Thus there exists $c_{\Lambda,\kappa}$ such that:

$$\eta^W(i-j,i)\|[A^{i-j,j}]^{-1}\xi\| \leq e^{c_{\Lambda,\kappa}+c_{\Lambda,\kappa}C+c_{\Lambda,\kappa}\epsilon i-(\kappa/3)j}.$$

From this, the claim now follows by summing the series that defines $\|\cdot\|_i'$ on W_i. ☐

We now finish checking item (3). The lower bound in (2.10) holds because $\|\xi_V\|_i \leq \|\xi_V\|_i'$ and $\|\xi_W\|_i \leq \|\xi_W\|_i'$ from the definitions. For the upper bound, we use Claims 2.6 and 2.5 giving the reverse comparison of the norms $\|\cdot\|$ and $\|\cdot\|_i'$ restricted to V_i and W_i. If we decompose $\xi \in E_i$ as a sum of $\xi_V \in V_i$ and $\xi_W \in W_i$, then

$$\|\xi\|_i' \leq \|\xi_V\|_i' + \|\xi_W\|_i' \leq e^{C_{\Lambda,\kappa}\epsilon i+C_{\Lambda,\kappa}C+C_{\Lambda,\kappa}}(\|\xi_V\|_i + \|\xi_W\|_i).$$

Then using that the angle θ between V_i and W_i is at least $e^{-C}e^{-\epsilon i}$ by assumption, we obtain the upper bound in (2.10) as it follows that for $* \in \{V, W\}$ we have $\|\xi_*\| \leq 2\|\xi\|_i/\theta$. ☐

The following lemma will be useful to bound the norms of the derivatives in adapted charts.

Lemma 2.7. (a) Fix $c, \Lambda > 0$. Then there exists $C_{c,\Lambda} > 0$ such that the following holds:

Suppose that A_n is a sequence of linear maps $A_n\colon \mathbb{R}^n \to \mathbb{R}^n$ and η_i is a sequence of numbers as in Lemma 2.1. Suppose that $c^{-1} \leq \eta_i \leq c$ and $e^{-\Lambda} \leq m(A) \leq \|A\| \leq e^{\Lambda}$. Let $\|\cdot\|_i^*$ be either the forwards-looking or backwards-looking metric from Lemma 2.1. Then for any $\|\cdot\|_i^*$ unit vector ξ,

$$C_{c,\Lambda}^{-1} \leq \|A_i\xi\|_{i+1}^* \leq C_{c,\Lambda}.$$

(b) In particular, given $d, C, \lambda, \kappa, \epsilon$ there is a constant $\mathfrak{C}$ such that under the assumption of Lemma 2.3 we have that

$$\mathfrak{C}^{-1}\|\xi\|_i' \leq \|A_i\xi\|_{i+1}' \leq \mathfrak{C}\|\xi\|_i'$$

where $\|\cdot\|'$ is the adapted metric constructed in that Lemma.

Proof. Part (a) is immediate from (2.6).

By part (a), to prove part (b) it suffice to show that the η_i^V and η_i^W defined in (2.16) and (2.17) are uniformly bounded from above and below. To this end it is enough to check that the term $\|A^{i_k,t_k}|_{V_{i_k}}\|''^{1/t_k}$ is uniformly bounded. Since the sequence A_i is uniformly bounded by assumption, the needed conclusion follows by another application of (2.6). □

3. Step 2. Construction of adapted charts

Now, using the adapted metric from Lemma 2.3, we construct adapted charts. This step is standard, see for example Barreira-Pesin [BP07, Thm. 5.6.1] or Brown [Bro22, Sec. 4].

Adapted charts are defined by composing the exponential map on M with a linear map $C_n\colon \mathbb{R}^d \to T_{f^n(x)}M$, where the linear map is chosen so that it pushes forward the Euclidean metric on $\mathbb{R}^n$ to the adapted metric on $T_{f^n(x)}M$. Namely, we require that for any vectors $v_1, v_2 \in \mathbb{R}^n$

$$\langle C_n v_1, C_n v_2\rangle_{f^n(x)} = \langle v_1, v_2\rangle_n'.$$

Then we construct adapted charts $\widehat{\Phi}_n\colon \mathbb{R}^d \to M$ by defining:

$$\widehat{\Phi}_n(v) = \exp_{f^n(x)}(C_n(v)).$$

One can then examine the map f_n using these charts. Consider the following map, whose domain of definition must be suitably restricted:

$$\breve{f}_n = \widehat{\Phi}_{n+1}^{-1} \circ f_n \circ \widehat{\Phi}_n.$$

By construction, $D_0\breve{f}_n$ has exactly the properties of $D_{f^{n-1}(x)}f_n$ when viewed in the adapted metric.

To ensure that within this chart the C^k norm of $\breve{f}_n - D_x\breve{f}_n$ is small, we modify our charts $\widehat{\Phi}_n$ by composing with an additional (tempered) contraction r_n, i.e. we define $\Phi_n(v) = \widehat{\Phi}_n(r_n v)$. Then we define

$$\tilde{f}_n = \Phi_{n+1}^{-1} \circ f_n \circ \Phi_n.$$

Due to this rescaling, we can ensure that the C^k norm of $\tilde{f}_n - D_0\tilde{f}_n$ is arbitrarily small.

Let us describe the choice of r_n. Observe that if we have a function g, then the function $\lambda_1^{-1}g(\lambda_2 x)$ has its kth derivatives bounded by $\lambda_1^{-1}\lambda_2^k\|g\|_k$, where here, as elsewhere, we write $\|g\|_k$ for the seminorm given by the supremum of the kth order partial derivatives of g. Hence $r_{n+1}^{-1}\tilde{f}_n(r_n x)$ will have its kth derivatives bounded by $r_{n+1}^{-1}r_n^k\|\tilde{f}_n\|_k$. Because the adapted metrics themselves are tempered, there exists some m_1 such that $\|\breve{f}_n\|_k \le e^{C_1} e^{m_1 \epsilon n}$, where C_1 is a constant depending on $(C, \kappa, \epsilon, \Lambda, \kappa, d, \ell)$ and m_1 is given by the constant $c_0 = C_{\Lambda,\kappa,d,\ell}$ in Lemma 2.3. Thus, if we want to make the kth derivatives arbitrarily small, we can let

$$r_n = e^{-D} e^{-m_1 \epsilon n} \tag{3.1}$$

where D is a large constant depending only on (C, κ, ϵ). Note that by further increasing D, we can make the ϵ_1 in the following lemma arbitrarily small, but this does *not* affect the singular values of $D\tilde{f}_n$ as changing D dilates all of the adapted charts Φ_n by the same constant.

This construction and the subsequent estimates are standard. Working through them yields the following lemma.

Lemma 3.1. (Adapted Charts) Fix dimensions d, d_{ss}, d_c and $\kappa > 0$. Then for any $\Lambda > 0$ there exists $m = m(\Lambda, \kappa, d_{ss}, d_c) \in \mathbb{N}$ such that for all sufficiently small $\epsilon > 0$ and any $C > 0$, there exist $\sigma_1, \sigma_2 > 0$ such that for all $\epsilon_1 > 0$, there exists $\hat{C}$ such that the following holds.

Assume that $D_x f^n$ has a (C, κ, ϵ)-subtempered dominated splitting of index ℓ with a sequence of associated subspaces $T_{f^n(x)}M = V_{n+1} \oplus W_{n+1}$, and e^{Λ} is an upper bound on the C^{k-1} norm and conorm of $D_x f^n$. Then there is a sequence $\{\Phi_n\}$ of C^∞ embeddings of a d-dimensional disk onto a neighborhood of $f^{n-1}(x)$, such that if r_n is defined as in (3.1) and $\ell(n) := r_{n+1}^{-1} r_n^{-k}$, then

1. $\Phi_n \colon B(0,1) \to M$ with $\Phi_{n+1}(0) = f^n(x)$.
2. $D_0 \Phi_n \mathbb{R}^{d_{ss}} = V_n$ and $D_0 \Phi_n \mathbb{R}^{d_c} = W_n$.
3. The map $\tilde{f}_n = \Phi_{n+1}^{-1} \circ f_n \circ \Phi_n$ is well defined on the ball $B(0, e^{-\hat{C}})$.
4. $D_0 \tilde{f}_n \mathbb{R}^{d_{ss}} = \mathbb{R}^{d_{ss}}$, and $D_0 \tilde{f}_n \mathbb{R}^{d_c} = \mathbb{R}^{d_c}$. For unit vectors $v \in \mathbb{R}^{d_{ss}}$ and $w \in \mathbb{R}^{d_c}$,
$$\sigma_1 \le \|D_0 \tilde{f}_n v\| \le e^{\kappa/20} \|D_0 \tilde{f}_n w\| \le \sigma_2.$$
5. $\|D\tilde{f}_n - D_0 \tilde{f}_n\|_{C^{k-1}} \le \epsilon_1$.
6. For every $x, y \in B(0,1)$, $\quad \max\{\|D\Phi_n^{-1}\|_{C^{k-1}}, \|D\Phi_n\|_{C^{k-1}}\} \le \ell(n)$.
7. ($\ell(n)$ is slowly growing) $\ell(1) \le e^{\hat{C}}$ and $e^{-\epsilon m} \le \ell(n)\ell(n+1)^{-1} \le e^{\epsilon m}$.

Moreover, the analogous claims hold for $\tilde{f}_n^{-1}$.

4. Step 3. Extending dynamics in Lyapunov charts to globalized dynamics

Using the adapted coordinates, we now construct the globalized coordinates to which we will apply a graph transform argument described below.

By using a bump function ϕ supported in the ball $B(0, e^{-\hat{C} - m\epsilon})$, we can can take the maps $\tilde{f}_n$ from Lemma 3.1 and define a new map

$$\hat{f}_n = D_0 \tilde{f}_n + (1 - \phi(x))[\tilde{f}_n - D_0 \tilde{f}_n].$$

The important claim about the globalized dynamics is the following:

Lemma 4.1. If x has a (C,κ,ϵ)-codimension ℓ subtempered trajectory as in Lemma 3.1, then for any $\epsilon_1 > 0$ (after possibly decreasing the constant $\hat{C}$ in Lemma 3.1) there exist C^k diffeomorphisms $\hat{f}_n\colon \mathbb{R}^d \to \mathbb{R}^d$ such that:

(1) $\hat{f}_n$ agree with the diffeomorphisms $\tilde{f}_n$ from Lemma 3.1 on the ball of radius $B(0, e^{-\hat{C}-m\epsilon}/2)$.
(2) $\|D\hat{f}_n - D_0\hat{f}_n\|_{C^{k-1}} \le \epsilon_1$.

The same claims hold for $\hat{f}_n^{-1}$, *mutatis mutandis.*

5. Step 4. One step graph transform

In this section we estimate the norm of the graph transform in $\mathbb{R}^d$ under the assumption of a gap in the singular values. This section contains two families of estimates depending on the regularity of the dynamics and the graph being transformed. The first estimate is for Lipschitz graphs, and the second estimate is on C^k graphs. The former estimates will be useful for showing that a stable curve actually exists, and the latter will be used *post hoc* to show that the stable curve thus constructed is in fact smooth. Note that in the proof of the main result our estimates will be applied to the maps $(\hat{f}_n)^{-1}$ where $\hat{f}_n$ is obtained from Lemma 4.1.

Remark 5.1. We do not actually assume at this step that the unstable direction is expanding—in the case $m(A)<1$, we just estimate how much worse a graph may get under the graph transform. For example, consider what happens if A and B are conformal matrices of norms 1/2 and 1/3.

5.1. **Norms of compositions.** Below we will be careful in our handling of the norms because we will consider functions with unbounded domains and images. We will define some semi-norms on functions $\phi\colon \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$. If α is a multindex, such as indexes a mixed-partial derivative, we write $|\alpha|$ for its order. Write π_i for projection onto the ith coordinate. First we define the semi-norm

$$|\phi|_k = \sup_{x\in\mathbb{R}^{d_1}, |\alpha|=k, 1\le i\le d_2} |\partial^\alpha \pi_i \phi|_k .$$

Note that this is a semi-norm as we only look at the kth order derivatives.

We let $\|\phi\|_k$ denote the usual C^k-norm, i.e. the sum of the C^0 norm and $|\phi|_k$. Recall that these norms are log convex, and moreover that for two functions fg,

$$\|fg\|_{C^k} \le C_k(\|f\|_0\|g\|_k + \|f\|_k\|g\|_0), \tag{5.1}$$

see e.g. [Hör76, App. A] for a further discussion of this and similar estimates.

The following estimate is a slight variant of [Hör76, Lem. A.8],

$$\|f\circ g\|_k \le C_k(\|f\|_k\|g\|_1^k + \|f\|_1\|g\|_k + \|f\|_0). \tag{5.2}$$

In the following lemma, we give a version of this estimate adapted to functions $\mathbb{R}^d \to \mathbb{R}^d$ that have unbounded image but bounded derivatives.

Lemma 5.2. Suppose that $k \in \mathbb{N}$ and fix dimensions d_1, d_2, d_3. Then there exists C_k such that if $g\colon \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ and $f\colon \mathbb{R}^{d_2} \to \mathbb{R}^{d_3}$ are two C^k functions defined on open domains so that $f \circ g$ is well defined then

$$\|f \circ g\|_k \leq C_k(\|f\|_k \|g'\|_0^k + \|f'\|_0 \|g'\|_{k-1} + \|f\|_0), \tag{5.3}$$

where f' and g' represent the Jacobi matrices of f and g.

Proof. The proof is a straightforward induction using the more standard estimate (5.2) above. For $k = 1$, the estimate is immediate from the chain rule. We proceed by induction.

Suppose that $k > 1$. Note that by (5.1),

$$|f \circ g|_k = |f' \circ g g'|_{k-1} \leq C_k(\|f' \circ g\|_{k-1} \|g'\|_0 + \|f' \circ g\|_0 \|g'\|_{k-1}).$$

Applying our inductive hypothesis to the $\|f' \circ g\|_{k-1}$ term gives

$$|f \circ g|_k \leq C([\|f'\|_{k-1} \|g'\|_0^{k-1} + \|f'\|_0 \|g'\|_{k-2} + \|f'\|_0] \, \|g'\|_0 + \|f'\|_0 \|g'\|_{k-1}).$$

All terms in the above expression are of the desired form except $\|f'\|_0 \|g'\|_{k-2} \|g'\|_0$. For this term, we interpolate $\|g'\|_{k-2} \leq C_k' \|g'\|_{k-1}^{\frac{k-2}{k-1}} \|g'\|_0^{\frac{1}{k-1}}$. Thus

$$\|f'\|_0 \|g'\|_{k-2} \|g'\|_0 \leq C_k' \|f'\|_0 (\|g'\|_{k-1})^{\frac{k-2}{k-1}} (\|g'\|_0^k)^{\frac{1}{k-1}}.$$

Applying the Young inequality with $p = \frac{k-1}{k-2}$, $q = k - 1$ to the last two terms, we see that

$$\|f'\|_0 \|g'\|_{k-2} \|g'\|_0 \leq C_k'' \|f'\|_0 \left[\|g'\|_{k-1} + \|g'\|_0^k\right],$$

each term of which has the desired form, which concludes the induction. $\square$

5.2. **Main results.** Below we use the $*$-metric: for maps $\phi_1, \phi_2 \colon \mathbb{R}^{d_{uu}} \to \mathbb{R}^{d_c}$ with $\phi(0) = 0$, define:

$$d^*(\phi_1, \phi_2) = \sup_{x \in \mathbb{R}^{d_{uu}} \setminus \{0\}} \frac{\|\phi_1(x) - \phi_2(x)\|}{\|x\|}. \tag{5.4}$$

This metric is used in [BP07] and [Shu87]. Note that d^*–distance between Lipschitz graphs is finite.

Given $\phi\colon \mathbb{R}^{d_{uu}} \to \mathbb{R}^{d_c}$ the graph of ϕ is the set

$$\mathcal{G}_\phi = \{(x, \phi(x)) : x \in \mathbb{R}^{d_{uu}}\} \subset \mathbb{R}^d.$$

If $F\colon \mathbb{R}^d \to \mathbb{R}^d$ is such that the set $F(\mathcal{G}_\phi)$ is also be a graph of some function $\tilde{\phi}$ so that $\mathcal{G}_{\tilde{\phi}} = F(\mathcal{G}_\phi)$ for some function $\tilde{\phi}$, then we call $\tilde{\phi}$ a *graph transform of* ϕ.

Proposition 5.3. Fix $\kappa > 0$ and $d_{uu}, d_c \in \mathbb{N}$.

Let $F\colon \mathbb{R}^{d_{uu}} \oplus \mathbb{R}^{d_c} \to \mathbb{R}^{d_{uu}} \oplus \mathbb{R}^{d_c}$ be a C^1 diffeomorphism of the form

$$(x, y) \mapsto (Ax + f_1(x, y), By + f_2(x, y)),$$

where A and B are linear maps satisfying $m(A) > \|B\| e^\kappa$ and $f_j(x, y)$ vanish at the origin together with their derivatives. Write F^{-1} in the form

$$(x, y) \mapsto (A^{-1}x + g_1(x, y), B^{-1}y + g_2(x, y)).$$

Suppose that $\phi\colon \mathbb{R}^{d_{uu}} \to \mathbb{R}^{d_c}$ defines a graph over $\mathbb{R}^{d_{uu}}$ with $\phi(0) = 0$. Let

$$\max\{\|f_1\|_{C^1}, \|g_1\|_{C^1}\} = \epsilon_1, \quad \max\{\|f_2\|_{C^1}, \|g_2\|_{C^1}\} = \epsilon_2.$$

Then as long as

$$m(A) - \epsilon_1 - \epsilon_1 \operatorname{Lip}\phi > 0, \tag{5.5}$$

the graph transform of ϕ is well defined; we write it $\tilde{\phi}\colon \mathbb{R}^{d_{uu}} \to \mathbb{R}^{d_c}$.

(1) If ϕ satisfies (5.5), then

$$\operatorname{Lip}(\tilde{\phi}) \le \frac{\|B\| \operatorname{Lip}(\phi) + \epsilon_2(1 + \operatorname{Lip}(\phi))}{m(A) - \epsilon_1 - \epsilon_1 \operatorname{Lip}\phi}. \tag{5.6}$$

(1*) If ϕ is C^1, then (1) holds with $\|\phi\|_{C^1}$ in the place of $\operatorname{Lip}(\phi)$.

(2) If $\phi_1, \phi_2\colon \mathbb{R}^{d_{uu}} \to \mathbb{R}^{d_c}$ satisfy (5.5) and

$$\max\{\operatorname{Lip}\phi_1, \operatorname{Lip}\phi_2, \operatorname{Lip}\tilde{\phi}_1, \operatorname{Lip}\widetilde{\phi_2}\} \le \gamma, \tag{5.7}$$

then

$$d^*(\tilde{\phi}_1, \widetilde{\phi_2}) < \frac{\|B\| + \epsilon_2 + \gamma\epsilon_1}{m(A) - \epsilon_1 - \epsilon_1\gamma} d^*(\phi_1, \phi_2). \tag{5.8}$$

(3) If $\tilde{\phi}$ is the graph transform of ϕ, then

$$\|\pi_1(F^{-1}(x_1, \tilde{\phi}_1(x))) - \pi_1(F^{-1}(x_2, \tilde{\phi}(x_2)))\| \le (m(A)^{-1} + \epsilon_1 + \epsilon_1 \operatorname{Lip}\tilde{\phi})\|x_1 - x_2\|.$$

Under additional assumptions on F and ϕ, we can control the C^k norm of the graph transform.

Proposition 5.4. Suppose that in addition to assumptions of Proposition 5.3 we also have

$$\|f_1\|_{C^k} \le \epsilon_1, \quad \|f_2\|_{C^k} \le \epsilon_2, \tag{5.9}$$

and that there exists $\Lambda > 0$ such that

$$e^{-\Lambda} \le m(B) \le \|B\| \le m(A) \le \|A\| \le e^{\Lambda}. \tag{5.10}$$

Also suppose that there exists some $D_0 > 0$ such that:

$$\|\phi\|_{C^1} \le D_0, \quad \|\tilde{\phi}\|_{C^1} \le D_0, \tag{5.11}$$

and that there is $0 \le \epsilon < 0.01$ such that

$$(m(A) - \epsilon_1 - \epsilon_1\|\phi\|_{C^1})^{-1} \le m(A)^{-1}(1 + \epsilon), \tag{5.12}$$

Fixing the dimensions, there are constants C_k, $C_{\Lambda,D_0,k}$ and $C_{\Lambda,k}$ such that: If ϕ is a C^k graph that satisfies (5.11) and (5.12), then

$$\|\tilde{\phi}\|_{C^k} \le C_k \frac{\|B\|}{m(A)^k}\|\phi\|_{C^k} + C_{\Lambda,D_0}\epsilon_2(1 + \|\phi\|_{C^k}) + C_{k,\Lambda}\|\phi\|_{C^0}. \tag{5.13}$$

5.3. **Lipshitz estimates.**

Proof of Proposition 5.3. To begin, we will check that as long as (5.5) is satisfied that the graph transform is well defined. For this we first study the map $\psi_\phi\colon \mathbb{R}^{d_{uu}} \to \mathbb{R}^{d_{uu}}$ giving the first coordinate of $F^{-1}(x, \phi(x))$.

Given a graph ϕ, define the function ψ_ϕ^{-1} via:

$$\psi_\phi^{-1}(x) := Ax + f_1(x, \phi(x)). \tag{5.14}$$

Note that $\psi_\phi^{-1}(0) = 0$ and that for any two points x, y,

$$\|\psi_\phi^{-1}(x) - \psi_\phi^{-1}(y)\| \geq [m(A) - \epsilon_1 - \epsilon_1 \operatorname{Lip}\phi]\|x - y\|. \tag{5.15}$$

Hence the function ψ_ϕ, defined as the inverse of ψ_ϕ^{-1} is well defined as long as (5.5) holds. Moreover, it follows from the Clarke inverse function theorem, which implies the inverse function theorem for Lipshitz functions, that

$$\operatorname{Lip}(\psi_\phi) \leq (m(A) - \epsilon_1 - \epsilon_1 \operatorname{Lip}\phi)^{-1}. \tag{5.16}$$

Thus we see that if $\phi\colon \mathbb{R}^{d_{uu}} \to \mathbb{R}^{d_c}$ satisfies (5.5), then its graph transform $\tilde{\phi}$ is well defined and is given by the following formula:

$$\tilde{\phi}(x) = B\phi(\psi_\phi(x)) + f_2(\psi_\phi(x), \phi(\psi_\phi(x))). \tag{5.17}$$

Combining (5.17) with (5.16) gives (5.6) proving item (1). Next we establish item (2).

Before we proceed, note that by (5.14)

$$\|\psi_{\phi_1}^{-1}(x) - \psi_{\phi_2}^{-1}(x)\| \leq \|f_1(x, \phi_1(x)) - f_1(x, \phi_2(x))\| \leq \epsilon_1\|\phi_1(x) - \phi_2(x)\|. \tag{5.18}$$

Next we estimate $\|\psi_{\phi_1}(x) - \psi_{\phi_2}(x)\|$. For this, note that if (x, y_1) and (x, y_2) are points with the same x-coordinate, then $\pi_1 F^{-1}(x, y)$ satisfies:

$$\|\pi_1 F^{-1}(x, y_1) - \pi_1 F^{-1}(x, y_2)\| \leq \epsilon_1\|y_1 - y_2\|. \tag{5.19}$$

Thus

$$\|\psi_{\phi_1}(x) - \psi_{\phi_2}(x)\| \leq \epsilon_2\|\tilde{\phi}_1(x) - \widetilde{\phi_2}(x)\|.$$

We now turn to the estimate on $\|\tilde{\phi}_1 - \widetilde{\phi_2}\|_*$. We will consider the difference between $\tilde{\phi}_1$ and $\tilde{\phi}_2$ at the point $\psi_{\phi_1}^{-1}(x)$:

$$\begin{aligned}\|\tilde{\phi}_1(\psi_{\phi_1}^{-1}(x)) - \tilde{\phi}_2(\psi_{\phi_1}^{-1}(x))\| &= \|\tilde{\phi}_1(\psi_{\phi_1}^{-1}(x)) - \tilde{\phi}_2 \circ \psi_{\phi_2}^{-1}(x)\| + \|\tilde{\phi}_2 \circ \psi_{\phi_2}^{-1}(x) - \tilde{\phi}_2 \circ \psi_{\phi_1}^{-1}(x)\| \\ &=: I + II.\end{aligned}$$

First, we estimate I. From the formula for the graph transform (5.17),

$$\begin{aligned} I &\leq \|B\phi_1(x) - B\phi_2(x)\| + \|f_2(x, \phi_1(x)) - f_2(x, \phi_2(x))\| \\ &\leq \|B\|\|\phi_1(x) - \phi_2(x)\| + \epsilon_2\|\phi_1(x) - \phi_2(x)\| \end{aligned}$$

Next, by (5.18), $II \leq \operatorname{Lip}(\tilde{\phi}_2)\epsilon_1\|\phi_1(x) - \phi_2(x)\|$.

We now estimate $d^*(\tilde\phi_1, \tilde\phi_2)$:

$$\frac{\|\tilde\phi_1(\psi_{\phi_1}^{-1}(x)) - \tilde\phi_2(\psi_{\phi_1}^{-1}(x))\|}{\|\psi_{\phi_1}^{-1}(x)\|} \leq$$

$$\frac{\|B\|\|\phi_1(x) - \phi_2(x)\| + \epsilon_2\|\phi_1(x) - \phi_2(x)\| + \mathrm{Lip}(\tilde\phi_2)\epsilon_1\|\phi_1(x) - \phi_2(x)\|}{\|\psi_{\phi_1}^{-1}(x)\|}$$

Because $\psi_\phi(0) = 0$ and (5.15), the right hand side is bounded by

$$\frac{(\|B\| + \epsilon_2 + \mathrm{Lip}(\tilde\phi_2)\epsilon_1)\|\phi_1(x) - \phi_2(x)\|}{(m(A) - \epsilon - \epsilon\,\mathrm{Lip}\,\phi_1)\|x\|}.$$

Now (5.7) gives $d^*(\tilde\phi_1, \tilde\phi_2) \leq \dfrac{\|B\| + \epsilon_2 + \gamma\epsilon_1}{m(A) - \epsilon_1 - \epsilon_1\gamma} d^*(\phi_1, \phi_2)$. completing the proof of item (2).

Next, we estimate how much F^{-1} contracts distance in the first factor to verify item (3). The argument is similar to above, but we want to have $(\tilde\phi)$ on the right hand side.

$$\|\pi_1(F^{-1}(x_1, \tilde\phi(x_1))) - \pi(F^{-1}(x_2, \tilde\phi(x_2)))\| \leq \|A^{-1}(x_1 - x_2) + f_1^{-1}(x_1, \tilde\phi(x_1)) - f_1^{-1}(x_2, \tilde\phi(x_2))\|$$

$$\leq m(A)^{-1}\|x_1 - x_2\| + \epsilon_1\|x_1 - x_2\| + \epsilon_1\,\mathrm{Lip}\,\tilde\phi\|x_1 - x_2\| \leq (m(A)^{-1} + \epsilon_1 + \epsilon_1\,\mathrm{Lip}(\tilde\phi))\|x_1 - x_2\|,$$

which is the estimate in item (3). □

5.4. C^k estimates.

Proof of Proposition 5.4. We need to estimate the C^k norm of (5.17). We can do this using the chain rule once we have a suitable estimate on the derivatives of ψ_ϕ.

Estimates of the norms of ψ_ϕ. First we estimate $\|\psi'_\phi\|_0$. From (5.15) and (5.12), we immediately have the upper bound

$$\|\psi'_\phi\|_0 \leq (m(A) - \epsilon_1 - \epsilon_1\,\mathrm{Lip}(\phi))^{-1} < m(A)^{-1}(1 + \epsilon). \tag{5.20}$$

Next we obtain an estimate on the C^k norm of ψ_ϕ. We will first use a fact that if a C^k diffeomorphism f is C^1 close to the identity, then the C^k norm of f and f^{-1} are comparable. Specifically, for any j, there exists $\epsilon_0 > 0$ and some constant C_j such that if $G\colon \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism of the form $x \mapsto x + X(x)$, where $\|X\|_{C^1} < \epsilon_0$ where ϵ_0 is a fixed small number, then G^{-1} may be written in the form:

$$G^{-1}(X) = x + X_{-1}(x),$$

and

$$\|X_{-1}\|_{C^j} \leq C_j\|X\|_{C^j}. \tag{5.21}$$

For a proof of this fact, see [DeW24, Lem. 48], see also [Ham82, Lem. 2.3.6].

To apply this estimate, denote

$$\widetilde{\psi_\phi^{-1}} = A^{-1}\psi_\phi^{-1} = x + A^{-1}f_2(x, \phi(x)).$$

Then by (5.21) $\widetilde{\psi_\phi} = (\widetilde{\psi_\phi^{-1}})^{-1} = x + \widetilde{Q}(x)$, where

$$\|\widetilde{Q}\|_{C^k} \leq C_k \|A^{-1} f_2(x, \phi(x))\|_{C^k}.$$

To conclude, it suffices to estimate $\|f_2(x, \phi(x))\|_{C^k}$. By Lemma 5.2,

$$\begin{aligned}\|f_2 \circ (x, \phi(x))\|_{C^k} &\leq C(\|f_2\|_{C^k} \|(x, \phi(x))'\|_0^k + \|f_2'\|_0 \|(x, \phi(x))'\|_{k-1} + \|f_2\|_0) \\ &\leq C(\epsilon_2(1 + D_0)^k + \epsilon_2(1 + \|\phi\|_{C^k} + \epsilon_2)) \leq C'\epsilon_2(1 + \|\phi\|_{C^k}).\end{aligned}$$

Thus as $\psi_\phi = (\widetilde{\psi_\phi^{-1}})^{-1} \circ A^{-1} = A^{-1}x + \widetilde{Q} \circ A^{-1}$, we obtain

$$\|\psi_\phi'\|_{k-1} \leq C_{\Lambda,k,D_0} \epsilon_2 (1 + \|\phi\|_{C^k}). \tag{5.22}$$

Estimate on norm of $\tilde{\phi}$. By definition,

$$\tilde{\phi}(x) = B\phi(\psi_\phi(x)) + f_2(\psi_\phi(x)), \phi(\psi_\phi(x))) =: I + II.$$

First we estimate term I. For this, we apply Lemma 5.2 and (5.22):

$$\begin{aligned}\|\phi \circ \psi_\phi\|_{C^k} &\leq C_k(\|\phi\|_{C^k} \|\psi_\phi'\|_0^k + \|\phi'\|_0 \|\psi_\phi'\|_{k-1} + \|\phi\|_{C^0}) \\ &\leq C_k(\|\phi\|_{C^k} m(A)^{-k}(1+\epsilon)^k + C'_{\Lambda,k,D_0}\epsilon_2(1 + \|\phi\|_{C^k}) + \|\phi\|_{C^0}). \qquad (5.23)\end{aligned}$$

As B is linear and $\epsilon < .01$, it follows that:

$$\|B \circ \phi \circ \psi_\phi\|_{C^k} \leq C_k \|B\| m(A)^{-k} \|\phi\|_{C^k} + C''_{\Lambda,k,D_0} \epsilon_2 (1 + \|\phi\|_{C^k}) + C'_k \|\phi\|_{C^0}.$$

Next we estimate II. Again, by Lemma 5.2,

$$\|f_2(\psi_\phi(x), \phi \circ \psi_\phi(x))\|_{C^k} \leq$$

$$C_k(\|f_2\|_{C^k} \|(\psi_\phi, \phi \circ \psi_\phi(x))'\|_0^k + \|f_2'\|_0 \|(\psi_\phi, \phi \circ \psi_\phi)'\|_{k-1} + \|f_2\|_0)$$

So, by (5.20)

$$\mathit{II} \leq C_k[\epsilon_2[m(A)^{-1}(1+\epsilon) + (m(A)(1+\epsilon))^k] + \epsilon_2[\|\psi_\phi'\|_{k-1} + \|\phi \circ \psi_\phi\|_{C^k} + \epsilon_2])$$

We have (5.22) to estimate $\|\psi_\phi'\|_{k-1}$, and (5.23) to estimate $\|\phi \circ \psi_\phi\|_{C^k}$. Thus combining the estimates on I and II above we find that there exists C_k depending only on k, and C_{Λ,k,D_0} and $C_{\Lambda,k}$ such that:

$$\|\tilde{\phi}\|_{C^k} \leq C_k \|B\| m(A)^{-k} \|\phi\|_{C^k} + C_{\Lambda,k,D_0} \epsilon_2 (1 + \|\phi\|_{C^k}) + C_{\Lambda,k} \|\phi\|_{C^0}.$$ □

6. Step 5: Conclusion; Construction of stable manifolds.

Proof sketch of Theorem 1.3. The proof has three main steps. First, we argue that the strong stable manifolds may be exhibited as a fixed point in the space of sequences of Lipschitz graphs. Then we show that due to smoothing properties of the graph transform, the Lipshitz graphs thus constructed must be C^{k-1} because the set of C^k functions of bounded norm is preserved by the graph transform. Finally, we study the rate of attraction of nearby point in both the direction of strong stable manifold and the transverse direction using invariant cones.

Lipshitz Strong Stable Manifolds. The proof that the strong stable manifolds are Lipschitz is completely standard in view of Proposition 5.3.

We work in the globalized version of the dynamics in the adapted charts obtained from Lemma 4.1.

Consider a sequence of Lipschitz maps $\phi_n\colon \mathbb{R}^{d_{ss}} \to \mathbb{R}^{d_c}$ indexed by $n \in \mathbb{N}$. We can endow the space $\mathfrak{L}$ of such sequences of maps with the norm:

$$d^*((\phi_{1,n}),(\phi_{2,n})) = \sup_n d^*(\phi_{1,n},\phi_{2,n}),$$

where d^* is the metric introduced in (5.4). Note that for each γ the set $\mathfrak{L}_\gamma$ of sequences such that all the maps $\phi_{1,n}$ are uniformly γ-Lipschitz has bounded diameter.

A sequence of maps $\hat{f}_n\colon \mathbb{R}^d \to \mathbb{R}^d$ acts on $\mathfrak{L}_\gamma$ by the graph transform under $\hat{f}_n^{-1}$. Write F for this action, so that $F((\phi_n)) = (\tilde{\phi}_n)$, where $\tilde{\phi}_n$ is the graph transform of ϕ_{n+1} by $\hat{f}_n^{-1}$.

We now apply Proposition 5.3. Note that for any $\gamma > 0$, as long as ϵ_1 and ϵ_2 are sufficiently small, F leaves invariant the space of sequences of γ-Lipschitz graphs passing through 0 due to (5.6). Moreover, it follows from (5.8), that as long as ϵ_1 and ϵ_2 are sufficiently small relative to γ, F is also a contraction of $(\mathfrak{L}_\gamma, d^*)$. It is a standard fact that $\mathfrak{L}_\gamma$ is complete in the d^* metric (see Shub [Shu87, Lem. III.3, III.4]). Thus by the Banach fixed point theorem, there exists a sequence of γ-Lipshitz graphs (ϕ_n) that is invariant under F. This is the sequence of graphs of the local stable manifolds for the globalized dynamics.

Regularity. To prove that the strong stable manifold is C^{k-1} regular, we will start with the sequence $\phi_n\colon \mathbb{R}^{d_{ss}} \to \mathbb{R}^{d_c}$ where each $\phi_n = 0$. By the previous part of the proof, defining F as there, $F^n((\phi_n))$ is converging to the graph of the strong stable manifold. Thus if we can show that the C^k norm of this sequence stays bounded, it then follows by Arzela-Ascoli that the limiting graph is uniformly C^{k-1}.

Suppose that $D_x f^n$ is a (C,κ,ϵ)-tempered trajectory. Then in the adapted coordinates, the singular value gap is at least $e^{\kappa/8}$ by Lemma 2.3. Let C_k be the constant—depending only on k—from Proposition 5.4. Choose N_0 such that

$$e^{-[\kappa/8]N_0} > 2C_k.$$

To examine the C^k norm, we will apply this estimate to blocks of length N_0. From (5.13), we have for a C^k graph ϕ with a uniform bound D_0 on its C^1 norm, that its graph transform $\tilde{\phi}$ satisfies:

$$\|\tilde{\phi}\|_{C^k} \leq C_k \frac{\|B\|}{m(A)^k}\|\phi\|_{C^k} + C_{\Lambda,k,D_0}\epsilon_2(1+\|\phi\|_{C^k}) + C_{k,N_0\Lambda}\|\phi\|_{C^0}. \tag{6.1}$$

Note that we have specifically chosen to look at the iterate N_0, where the coefficient on the first term is less than 1/2. Hence applying this estimate to F^{N_0}, we see that as long as ϵ_2 is sufficiently small, which may be chosen independent of these constants, then

$$\|\tilde{\phi}\|_{C^k} \leq \frac{3}{4}\|\tilde{\phi}\|_{C^k} + C_{k,N_0\Lambda}\|\phi\|_{C^0}.$$

Applying the iterative procedure on the space of graphs as in the first part of the proof we see that if we start with smooth sequences then the C^k norms will stay uniformly bounded in terms of the constants above. Hence the sequence is precompact in the C^{k-1} norm and the conclusion follows.

Rate of attraction. We now verify item (3). Because the manifold we constructed is a γ-Lipschitz graph, by Proposition 5.3(3) we see that if $z', z'' \in \mathcal{G}(\phi_0)$ then their images in the adapted charts are approaching each other at the rate

$$\|D_x f^n|_{V_0}\| e^{m_1 \epsilon n}, \tag{6.2}$$

where m_1 is a fixed constant. We need such an estimate for the dynamics in the original manifold. To obtain this, it suffices to observe that on the ball of radius $B(0, e^{-\Lambda-4\epsilon})$ in the adapted chart at time 0, the dynamics is precisely the pullback of the dynamics from the manifold, and moreover, due to Proposition 5.3(3), this set is invariant under forward interation as long as γ is sufficiently small. Hence, passing from the adapted charts to the ambient manifold introduces an additional multiplicative error $O\left(e^{m_2 \epsilon n}\right)$ due to Lemma 3.1(6). This finishes the proof of item (3).

Note that item (3) also tells us that the points on our manifold always stay in the neighbourhood of the orbit where the globalized dynamics is the actual dynamics, and so our manifold is actually the stable manifold for the actual sequence of maps.

Rate of repulsion. We now show that if a point is not in the stable manifold then it cannot approach the trajectory of x any faster than predicted by the conorm of the derivative of $D_x f^n$ on W_0. We estimate the distance between a point and the stable manifold in the adapted charts by examining how much a line segment Γ_i between the point and the curve stretches when we apply the dynamics. After each iterate, we reset the curve to be straight again. In this way we inductively estimate how the distance changes to obtain the desired bound.

In the adapted charts, by Lemma 3.1, at any point in the chart, the differential of $\tilde{f}_n$, is a small perturbation of the derivative at $(0,0)$. Thus

$$D_0\tilde{f}_n + O(\epsilon_1) = D_{x,y}\tilde{f}_n = \begin{bmatrix} A_{0,0} & 0 \\ 0 & B_{0,0} \end{bmatrix} + O(\epsilon_1) \tag{6.3}$$

where ϵ_1 can be made arbitrarily small by shrinking the chart. Furthermore we know that $m(B_{0,0}) > e^{\kappa/20}\|A_{0,0}\|$ by that same lemma. Note that we can choose $O(\epsilon_1)$ arbitrarily small relative to the norm, conorm, and gap between the singular values of $A_{0,0}$ and $B_{0,0}$.

Suppose that (x_0, y_0) is an arbitrary point that does not lie on the strong stable manifold. Then there is a unique point $(x_0, \hat{y}_0)$ in the strong stable manifold. Let $(x_i, y_i) = \tilde{f}^i(x_0, y_0)$ and let $(x_i, \hat{y}_i)$ be the point in the corresponding image of the stable set with the same x_i coordinate. Let Γ_i be the straight, unit speed curve going from (x_i, y_i) to $(x_i, \hat{y}_i)$. Write v_i for the unit vector tangent to Γ_i.

We now estimate the distance between (x_i, y_i) and $(x_i, \hat{y}_i)$. Note that, by (6.3), the tangent to $\tilde{f}_i\gamma_i$ at every point is of the form

$$\begin{bmatrix} 0 \\ B_{0,0}v_i \end{bmatrix} + O(\epsilon_1).$$

Because v_i is a unit vector and the norm and conorm of $B_{0,0}$ are bounded above and below, we see that $\tilde{f}_i\Gamma_i$ is tangent to a narrow cone $H_{\gamma_i}(v_{i+1})$ around the 1-dimensional subspace spanned by the vector $[0, B_{0,0}v_i]^T$. Thus for any $\epsilon_3 > 0$, as long as $\epsilon_1 > 0$ is sufficiently small, it follows that:

$$d(\tilde{f}_i(x_i, y_i), \tilde{f}_i(x_i, \hat{y}_i)) \geq e^{-\epsilon_3} m(B_{0,0}) \operatorname{len}(\Gamma_i).$$

Because the stable set is uniformly smooth, the distance between $\tilde{f}_i(x_i, \hat{y}_i)$ and $(x_{i+1}, \hat{y}_{i+1})$, the actual point sitting below $\tilde{f}^{i+1}(x_0, y_0)$ on the stable manifold, is of order $\epsilon_1 \operatorname{len}(\Gamma_i)$. Hence for arbitrarily small $\epsilon_4 > 0$, as long as we have chosen ϵ_1 sufficiently small in the first place, we obtain inductively that

$$d(\tilde{f}^n(x_0, y_0), (x_n, \hat{y}_n) \geq e^{-n\epsilon_4} m(B^n_{0,0}) \|y_0\|. \tag{6.4}$$

But due to the temperedness of the change of metrics, it follows that

$$m(B^n_{0,0}) \geq C_0 m(D_x f^n|_{W_1}) e^{-m_1\epsilon n}, \tag{6.5}$$

for the same fixed m_1 from before and W_0 being the other subspace provided by the splitting. Hence $(x_n, \hat{y}_n)$ approaches $(0, 0)$ at rate $\|D_x f^n|_{E^{ss}}\| e^{m_1\epsilon n}$, whereas, by (6.4) and (6.5), it follows that $\tilde{f}^n((x_0, y_0))$ stays at least distance $C_0 e^{-\epsilon_4 n} e^{\kappa/20} \|D_x f^n|_{E^{ss}}\|$ away from $(x_n, \hat{y}_n)$. The conclusion now follows from the triangle inequality for the distance in the tempered charts. Undoing the charts and using the temperedness gives the needed conclusion in the ambient manifold. □

Remark 6.1. Let us explain why the inequality (6.1) does not give a uniform C^k bound independent of $\|f_n\|_{C^k}$. Note that arranging that the constants be this small requires further reducing ϵ_1, ϵ_2, which in turn reduces the size of the manifold where our estimates apply.

7. Application to disintegration of volume in Pesin Theory

7.1. **Random dynamical systems.** This section contains an application of our main result to random dynamical systems. In that context, there are many points satisfying the conclusion of Theorem 1.3. Thus for almost every realization of the noise there is a positive measure set laminated by strong stable manifolds and we show that this lamination is absolutely continuous with controlled disintegration.

We start with some background on random dynamics.

Given a Borel probability measure μ compactly supported on $G = \mathrm{Diff}^1(M)$, the shift map $T\colon \Omega \to \Omega$, $T(\omega_n)_{n\geq 1} = (\omega_{n+1})_{n\geq 1}$ preserves $\mathbb{P} = \mu^{\otimes \mathbb{N}}$; and the skew product

$$F : \Omega \times M \to \Omega \times M \qquad (\omega, x) \mapsto (T(\omega), f^1_\omega(x))$$

preserves $\mathbb{P} \otimes \mathrm{vol}$. The following proposition is from [LQ95, Chapter I, Theorem 3.2].

Proposition 7.1. There is a function $r\colon M \to \{1, \dots, d\}$ and a Borel subset $\Lambda_0 \subset \Omega \times M$ with $\mathbb{P} \otimes \mathrm{vol}(\Lambda_0) = 1$ such that $F(\Lambda_0) = \Lambda_0$, and for every $(\omega, x) \in \Lambda_0$ there exist a sequence of linear subspaces of $T_x M$

$$\{0\} = V^{(0)}(\omega, x) \subsetneq V^{(1)}(\omega, x) \subsetneq \cdots \subsetneq V^{(r(x))}(\omega, x) = T_x M,$$

with $\dim V^{(i)}(\omega, x)$ depending only on x, i, and numbers

$$\lambda^{(1)}(x) < \cdots < \lambda^{(r(x))}(x)$$

such that for all $\xi \in E^{(i)} := V^{(i)}(\omega, x) \setminus V^{(i-1)}(\omega, x)$, $1 \leq i \leq r(x)$

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_x f^n_\omega(\xi)\| = \lambda^{(i)}(x),$$

and for vol-a.e. $x \in M$, for every $1 \leq i \leq r(x)$.

Moreover, $r(x), \lambda^{(i)}(x), V^{(i)}(\omega, x)$ depend measurably on $(\omega, x) \in \Lambda_0$, and for every $(\omega, x) \in \Lambda_0$ and $1 \leq i \leq r(x)$

$$r(f^1_\omega(x)) = r(x),\ \lambda^{(i)}(f^1_\omega(x)) = \lambda^{(i)}(x),\ D_x f^1_\omega(V^{(i)}(\omega, x)) = V^{(i)}(F(\omega, x)).$$

Further, if $E_P = \sum_{i\in I_P} E^{(i)}$, $\quad E_Q = \sum_{j\in I_Q} E^{(j)}$ for disjoint subsets I_P, I_Q, then

$$\lim_{n\to\infty} \frac{1}{n} \log \angle(Df^n_\omega E_P, Df^n_\omega E_Q) = 0,$$

The above proposition shows that if ν is a stationary measure on M, then ν-a.e. $x \in M$ and $\mu^{\mathbb{N}}$ a.e. random trajectory x has non-uniformly hyperbolic splittings corresponding to whatever the Lyapunov exponents themselves predict. We call the trajectories satisfying the conclusions of the above theorem *Lyapunov regular*.

In this section, we will always consider Lyapunov regular trajectories which also have a (C, κ, ϵ)-dominated splitting. By Corollary 1.5 their stable manifolds are the classical ones and we can take advantage of additional facts from Pesin theory.

7.2. **Regularity of the disintegration in the presence of a dominated splitting.** First, we will recall a standard fact concerning the disintegration of an SRB measure along the random stable manifolds. The definitions in this section follow [LQ95]. The only difference is that we work with the stable manifolds rather than the unstable manifolds. Below, for a Riemannian submanifold N, we write vol_N for the volume induced on N by the pullback metric. Also, if $\mathcal{W}$ is a foliation, we write $J^{\mathcal{W}}f$ for the Jacobian of f as map $\mathcal{W}(x) \to \mathcal{W}(f(x)) \cap f(\mathcal{W}(x))$. We write $f \propto g$ when two functions are equal up to rescaling by a fixed constant, i.e. $f = Cg$.

Definition 7.2. [LQ95, p. VII.1.6] A measurable partition η of $\Omega^{\mathbb{N}} \times M$ is *subordinate* to the W^s manifolds if for $\mu^{\mathbb{N}} \times \mathrm{vol}$-a.e. (ω, x) $\eta_\omega(x) = \{y : (\omega, y) \in \eta(\omega, x)\} \subset W^s(\omega, x)$ and $\eta_\omega(x)$ contains an open neighborhood of x in $W^u(\omega, x)$.

Note that the above definition does not require us to fix an ergodic stationary measure first and that the dimension of the partition elements may not be constant. It turns out that the disintegration of volume along the stable foliation always has a simple formula for its density. The following statement is the specialization of [LQ95, Cor. 8.2] to volume. That corollary gives the given formula for the density of the disintegration of an SRB measure (a notion we do not need and do not define here). [LQ95, Cor. VII.1.2] asserts that volume is always an SRB measure, hence we may apply the aforementioned corollary to deduce Proposition 7.3 below. Note that this proposition does *not* require that volume is ergodic.

Proposition 7.3. Suppose that μ is a compactly supported measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ and η is a partition that is subordinate to the W^s-manifolds. Then for a. e. ω, the disintegration of vol along η_ω is absolutely continuous.

Let $\tilde{\rho}$ be the density of vol disintegrated along $\eta_\omega(x)$. Then $\tilde{\rho}$ agrees almost everywhere with a function ρ that satisfies: for all $y, z \in \eta_\omega(x)$

$$\frac{\rho(\omega, y)}{\rho(\omega, z)} = \prod_{n=1}^{\infty} \frac{J^s D_{f^{n-1}_\omega(z)} f_{\sigma^n(\omega)}}{J^s D_{f^{n-1}_\omega(y)} f_{\sigma^n(\omega)}}. \tag{7.1}$$

Moreover, ρ is log-Lipschitz on $\eta_\omega(x)$.

The next proposition does not require the points to be Lyapunov regular.

Proposition 7.4. Suppose that μ is a compactly supported measure on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$. Then there exists $\hat{C}'$ such that for any $(C, \kappa, \kappa', \epsilon)$-tempered trajectory (ω, x), and any the function

$$y \mapsto \prod_{n=1}^{\infty} \frac{J^{ss} D_{f^{n-1}_\omega(x)} f_{\sigma^n(\omega)}}{J^{ss} D_{f^{n-1}_\omega(y)} f_{\sigma^n(\omega)}}$$

is uniformly $\hat{C}'$ log-Lipschitz on $W^{ss}_{\hat{C}^{-1}}(\omega, x)$, the manifold constructed in Theorem 1.3

Proof. We use the graph transform arguments from the previous section. We can check the result by carrying out the analogous computation in the same adapted charts used to construct the strong stable manifolds. Write $\hat{J}^{ss}\hat{f}_i$ for the Jacobian computed in the adapted charts at the point corresponding to $f^i_\omega(x)$. Then if we let $\hat{G}_n$ be the function on the adapted chart at $f^n_\omega(x)$ that gives the ratio of the ℓ-dimensional volume in the Euclidean metric on the adapted chart with the ℓ-dimensional volume with respect to the pushforward of the ambient metric on M, then

$$\prod_{i=1}^{n} \frac{J^{ss}D_{f_\omega^{n-1}(y)}f_{\sigma^n(\omega)}}{J^{ss}D_{f_\omega^{n-1}(y')}f_{\sigma^n(\omega)}} = \frac{\hat{G}_0(y)}{\hat{G}_0(y')}\prod_{i=1}^{n}\left[\frac{\hat{J}^{ss}\hat{f}_{n-1}(y)}{\hat{J}^{ss}\hat{f}_{n-1}(y')}\right]\frac{\hat{G}_n(y)}{\hat{G}_n(y')}.$$

We will now show that this converges to a uniformly log-Hölder function. We divide the estimate into two parts. First we consider the term in the square brackets. In the adapted charts, the W^{ss} manifold is given by the graph of a C^2 function whose norm is uniformly bounded along the entire trajectory and furthermore, the C^k-norm of the globalized dynamics $\hat{f}_n$ is uniformly bounded independent of n. Thus a single term

$$\frac{\hat{J}^{ss}\hat{f}_{i-1}(y)}{\hat{J}^{ss}\hat{f}_{i-1}(y')}$$

is bounded by $D_1 d(\hat{f}^n(x), \hat{f}^n(y))$ for some constant D_1. But by (6.2), there exists some $\kappa'' > 0$, D_2 such that $d(\hat{f}^n(x), \hat{f}^n(y)) \le D_2 e^{-n\kappa''}$. Hence the middle term is converging uniformly and moreover there is some uniform D_3 such that its logarithm is bounded by $D_3 d(y, y')$. Hence the middle term is converging to a uniformly log-Lipschitz function.

The terms $\hat{G}_n(y)$ measure the ratio between the Euclidean volume of the tangent to W^{ss} at $\hat{f}^n(y)$ with respect to the pushforward of the ambient volume. Hence these functions $\hat{G}_n$ have their C^1 norm controlled precisely by the norm of the adapted metric relative to the original metric. From the construction of the adapted metric, Lemma 2.3, it follows that there are constants m depending only on the dimension and D_4 depending only on the parameters describing the splitting given in the proposition such that $\|D\hat{G}\| < D_4 e^{m\epsilon}$. As before, $d(\hat{f}^n(x), \hat{f}^n(y)) \le D_2 e^{-n\kappa''}$. Hence as long as ϵ is sufficiently small, it follows that $\frac{\hat{G}_n(y)}{\hat{G}_n(\hat{y})} \le D_4 e^{n(m\epsilon - \kappa'')} d(y, y')$. We can then pass to the limit and conclude. □

Next consider the strong stable lamination restricted to a small ball. We are only interested in the points in the lamination containing $(C, \kappa, \kappa', \epsilon)$-tempered trajectories. Fix a point $x \in M$ and small $\delta > 0$. For a word ω, we will consider the following measurable partition of M. Let $G(\omega, C, \kappa, \epsilon)$ be the set of $(C, \kappa, \kappa', \epsilon)$-subtempered trajectories that are also Lyapunov regular for the word ω. Let

$$\Lambda^{ss}(\omega,x,\delta)=\bigcup_{y\in G(\omega,C,\kappa,\epsilon)\cap B_\delta(x)} W^{ss}_{\delta_0}(\omega,x).$$

We partition $\Lambda^{ss}(x,\delta)$ into the set of strong stable manifolds appearing on the right hand side of the equation and write $\mathcal{L}^{ss}(\omega,x,\delta)$ for this measurable partition. (Note that this notation is suppressing the dependence on $C,\kappa,\kappa',\epsilon$.)

Proposition 7.5. Suppose that μ is a compactly supported measure on $\mathrm{Diff}^3_{\mathrm{vol}}(M)$ and fix constants $(C,\kappa,\kappa',\epsilon)$ as in the definition of subtempered dominated splitting of index ℓ. There exists $\delta_0>0$ and $\hat{C}>0$ such that the following holds. For all $x\in M$, $0<\delta\le\delta_0$, and a.e. ω, the disintegration of vol-along $\Lambda^{ss}(\omega,x,\delta)$ is absolutely continuous along $W^{ss}(\omega,x)$ with positive density ρ. Moreover, for all $z,z'\in W^{ss}(\omega,x)\cap B_\delta(x)$,

$$\left|\frac{\rho(z)}{\rho(z')}\right|\le\hat{C}.$$

Proof. Fix a measurable partition η subordinate to the W^s lamination of $\Omega^{\mathbb{N}}\times M$. From the main result of [Bro22], the subfoliation of the W^s lamination's leaves by the W^{ss} is always $C^{1+\text{Hölder}}$, and moreover its holonomies are $C^{1+\text{Hölder}}$ diffeomorphisms [Bro22, Thm. 2.2]. In particular, this subfoliation is absolutely continuous.

We compute disintegrations in two steps. If π is a partition refining η, then to compute the disintegration on an atom of π, we can first disintegrate along η, and then disintegrate those disintegrations along π.

As the W^{ss} leaves form subfoliation of the W^s-leaves, and the disintegration along the elements of η is absolutely continuous with the density ρ given by (7.1) in Proposition 7.3, it follows that if we disintegrate $\mathrm{vol}^{\eta_\omega(x)}$ along the W^{ss} subfoliation, that this further disintegration is also absolutely continuous and moreover its density is also log-Lipshitz. We would just like to know what this density is—for this we can iterate the dynamics and use the change of variables formula. Note that it suffices to do our computation only at times when $f^n_\omega(x)$ is in a leaf where the regularity of the W^{ss} foliation is controlled.

Fix some big $D>0$. Let $\Xi(D)$ be the set of (ω,x) such that

(1) (ω,x) is Lyapunov regular, so that $W^s(\omega,x)$ and $W^{ss}(\omega,x)$ are both defined.
(2) restricted to a ball of radius D^{-1} inside of $W^s_{D^{-1}}(\omega,x)$ the $C^{1+\text{Hölder}}$ norm of the W^{ss} foliation within $W^s_{D^{-1}}(\omega,x)$ is at most D and
(3) $\eta_\omega(x)$ contains a ball of radius $100D^{-1}$ inside of $W^s(\omega,x)$.

Note that as $D\to+\infty$, that the $\mu^{\mathbb{N}}\otimes\mathrm{vol}$ measure of $\Xi(D)$ goes to 1.

Let $\mathcal{L}^s$ be the lamination $\mathcal{L}^s=\bigcup_{z\in\Lambda} W^s_{100D^{-1}}(\omega,x)$, which is sublaminated by $\mathcal{L}^{ss}$.

We claim that if (ω, x) is a point, so that all of these disintegrations are defined along the orbit of (ω, x) and (ω, x) recurs infinitely often to Ξ^D, then

$$\frac{d\,\mathrm{vol}^{\mathcal{L}^{ss}(\omega,x)}(y)}{d\,\mathrm{vol}_{\mathcal{L}^{ss}(\omega,x)}(x)} = \prod_{n=1}^{\infty} \frac{J^{ss}D_{f_\omega^{n-1}(y)}f_{\sigma^n(\omega)}}{J^{ss}D_{f_\omega^{n-1}(x)}f_{\sigma^n(\omega)}}. \tag{7.2}$$

Note that for any atom of $\mathcal{L}^s$, for n a sufficiently large good time, then $f_\omega^n(\mathcal{L}^s(\omega, x))$ is an open subset of $\eta_{\sigma^n(\omega)}(f_\omega^n(x))$. Thus the conditional measure on $f_\omega^n(\mathcal{L}^s(\omega, x))$ is the restriction of the density to $f_\omega^n(\mathcal{L}^s(\omega, x))$. Furthermore, both $\mathcal{L}^s(\omega, x)$ and $f_\omega^n(\mathcal{L}^s(\omega, x))$ are subfoliated by W^{ss} leaves.

Due to the uniqueness of the disintegration along a measurable partition, the pushforward of the disintegration along a measurable partition is the disintegration of the pushward measure along the pushforward partition. Hence we can compute the density of the disintegration by pushing it forward under the dynamics. Let ξ be a measurable partition of M and let $f_*\xi$ be the pushforward of this partition. Given a volume preserving diffeomorphism $f\colon M \to M$ and a measurable partition ξ of M, then for almost every partition element $f_*\xi_x^{\mathrm{vol}}$ agrees with the disintegration $([f_*\xi])_x^{\mathrm{vol}}$.

Suppose that $n \in \mathbb{N}$ is some time such that $f_\omega^n(\mathcal{L}^s(\omega, x)) \subset \eta_{\sigma^n(\omega)}(x)$. Then due to the previous paragraph, the density of the disintegration along $f_\omega^n(\mathcal{L}^s(\omega, x))$ is proportional to the density along $\eta_{\sigma^n(\omega)}(f_\omega^n(x))$ and, moreover, the disintegration along $f_\omega^n(\mathcal{L}^{ss}(\omega, y))$ for $y \in \mathcal{L}^s$ is given by

$$(f_\omega^n)_*\,\mathrm{vol}^{\mathcal{L}^{ss}(\omega,y)} \propto \mathrm{vol}^{\mathcal{L}^{ss}(\sigma^n(\omega), f_\omega^n(y))}.$$

Write ρ_n for the density of the disintegration with respect to the Riemannian volume on $\mathcal{L}^{ss}(\sigma^n(\omega), f_\omega^n(y)$. Then

$$\rho_n(z) \propto \rho_0((f_\omega^n)^{-1}(z)) \left| J^{ss}_{(f_\omega^n)^{-1}(z)} f_\omega^n \right|. \tag{7.3}$$

In particular, this shows that for any $y, y' \in \mathcal{L}^{ss}(\omega, x)$ we have

$$\frac{\rho_0(y)}{\rho_0(y')} = \prod_{i=1}^{n} \frac{J^{ss}D_{f_\omega^{n-1}(y')}f_{\sigma^n(\omega)}}{J^{ss}D_{f_\omega^{n-1}(y)}f_{\sigma^n(\omega)}} \frac{\rho_n(f_\omega^n(y))}{\rho_n(f_\omega^n(y'))}.$$

We take a limit along a sequence of n_i of times where $f_\omega^{n_i}(\mathcal{L}^s(\omega, x)) \in \Xi(D)$. Then $\frac{\rho_{n_i}(y)}{\rho_{n_i}(y')}$ is uniformly bounded independent of x, y, y', ω due to item (2) in the definition of $\Xi(D)$, and we are able to conclude that (7.2) holds almost everywhere on the partition $\mathcal{L}^{ss}$.

The result now follows from Proposition 7.4. □

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